Session 9: Portfolio Theory V

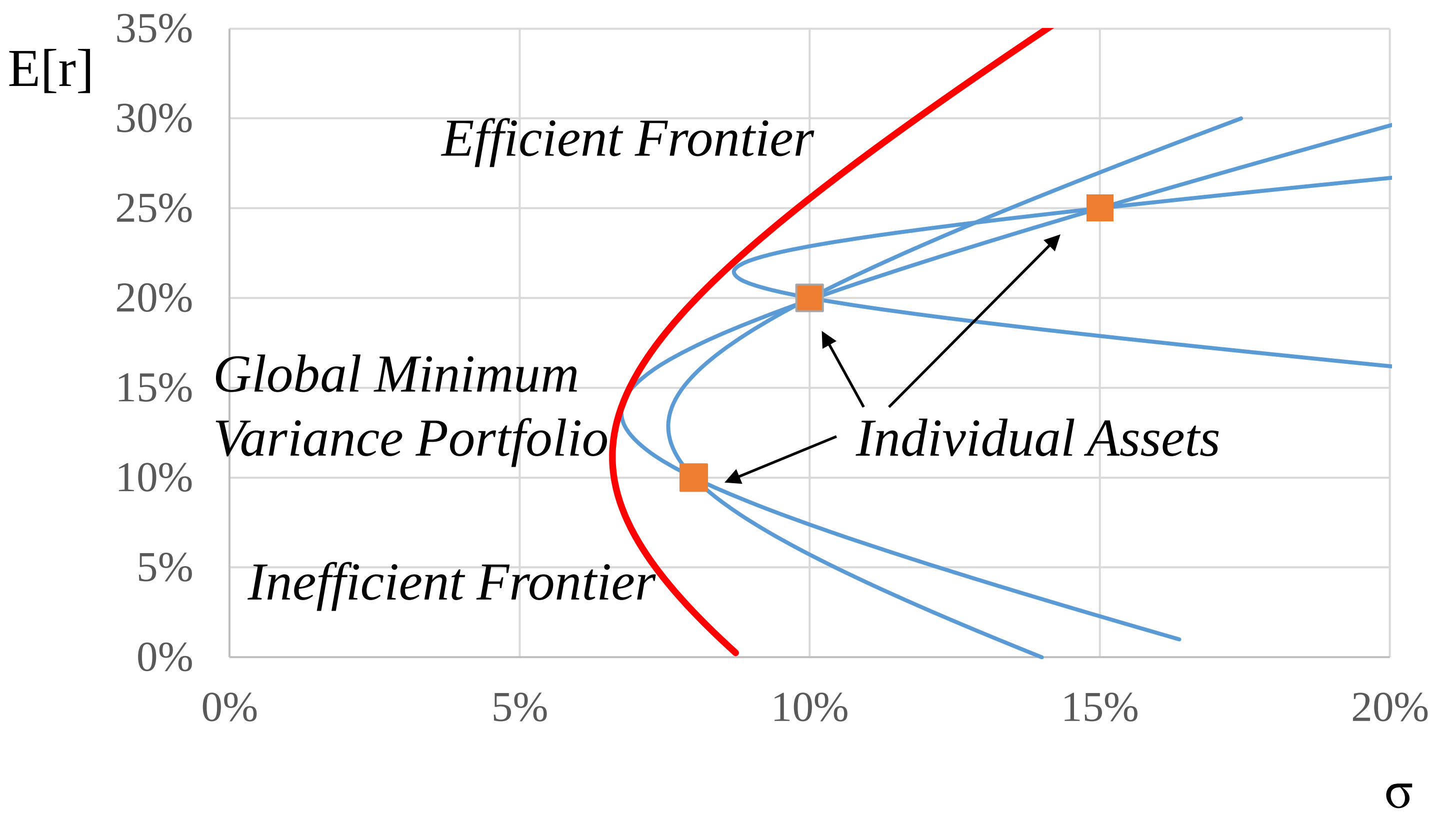
Fall 2025

Outline

- Portfolio selection with a risk-free and many risky securities
- Systematic and idiosyncratic risk
- The Single Index Model

Many Risky Assets

Investment opportunity set



Optimal Portfolio Selection

1. Create the set of possible mean-SD combinations from different portfolios of risky assets
2. Find the *tangency portfolio*, that is, the portfolio with the highest *Sharpe ratio*:

$$SR_p = \frac{E[r_p] - r_f}{\sigma_p}$$

3. Choose the combination of the tangency portfolio and the risk-free asset to suit your risk-return preferences

Two-Fund Separation

- All investors hold combinations of the same two "mutual funds":
  - *The risk-free asset*
  - *The tangency portfolio*
- An investor's risk aversion determines the fraction of wealth invested in the risk-free asset
- But, all investors should have the rest of their wealth invested in the tangency portfolio

Portfolio Optimizer

- Calculates optimal portfolio with 5 risky assets and 1 risk-free asset
- Why does optimal portfolio overload on 2?
- Why hold asset 4 at all?
- Importance of correlation:
  $\rho_{45} = 0 \rightarrow 0.7$
  $\rho_{45} = 0.7 \rightarrow 0.9$

Portfolio Variance and SD

- With 2 securities (N=2), the portfolio variance is

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

- With N securities, the portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j>i}^{N} w_i w_j \rho_{ij}\sigma_i\sigma_j$$

$$\sigma_p = \sqrt{\sigma_p^2}$$

"Matrix" Intuition

$$r_p = w_1 r_1 + \ldots + w_N r_N$$

$$\text{var}[r_p] = \text{cov}[r_p, r_p] = \text{cov}[w_1 r_1 + \ldots + w_N r_N, w_1 r_1 + \ldots + w_N r_N]$$

$$\text{cov}[w_i r_i, w_j r_j] = w_i w_j \rho_{ij} \sigma_i \sigma_j$$

	$w_1 r_1$	$w_2 r_2$	$\cdots$	$w_N r_N$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$w_1 w_2 \rho_{12} \sigma_1 \sigma_2$	$\cdots$	$w_1 w_N \rho_{1N} \sigma_1 \sigma_N$
$w_2 r_2$	$w_1 w_2 \rho_{12} \sigma_1 \sigma_2$	$w_2^2 \sigma_2^2$		
$\vdots$	$\vdots$		$\ddots$	
$w_N r_N$	$w_1 w_N \rho_{1N} \sigma_1 \sigma_N$			$w_N^2 \sigma_N^2$

Risk Reduction I

- Suppose we have an equally weighted portfolio ($w_i = 1/N$) of N independent stocks ($\rho_{ij} = 0$)
- The variance of the portfolio return is

$$\sigma_p^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 = \frac{1}{N}\left[\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2\right] = \frac{1}{N}[\text{average variance}]$$

- As the number of assets increase, the risk is completely diversified away

Risk Reduction II

- Suppose we have an equally weighted portfolio (w_i = 1/N) of N (correlated) stocks
- The variance of the portfolio return is:

$$\sigma_p^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sigma_i^2 + \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j>i}^{N}\operatorname{cov}(r_i, r_j)$$

$$= \frac{1}{N}\left[\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2\right] + \left[1 - \frac{1}{N}\right]\left[\frac{1}{N(N\text{-}1)/2}\sum_{i=1}^{N}\sum_{j>i}^{N}\operatorname{cov}(r_i, r_j)\right]$$

$$= \frac{1}{N}\begin{bmatrix}\text{average}\\ \text{variance}\end{bmatrix} + \left[1 - \frac{1}{N}\right]\begin{bmatrix}\text{average}\\ \text{covariance}\end{bmatrix}$$

Implications

What happens when N goes to infinity?

$$\sigma_p^2 = \frac{1}{N}\begin{bmatrix}\text{average} \\ \text{variance}\end{bmatrix} + \left[1 - \frac{1}{N}\right]\begin{bmatrix}\text{average} \\ \text{covariance}\end{bmatrix}$$

Variance of portfolio return → average covariance of returns

As the number of assets grows large you can get rid of a lot of the risk, but you can never get rid of the **covariance risk**: **non-diversifiable** or **systematic risk**.

The volatility of an individual asset is not a good indicator of its riskiness!

In fact, the only thing that matters is its **covariance** with other assets.

Classification of Risk

- The part that cannot be diversified away:
  - ✓covariance risk,
  - ✓systematic risk,
  - ✓non-diversifiable risk,

  e.g., market risk, macroeconomic risk

- The part that can be diversified away (in a large portfolio):
  - ✓idiosyncratic risk,
  - ✓non-systematic risk,
  - ✓diversifiable risk,
  - ✓unique risk,

  e.g., individual company risk

Systematic vs. Idiosyncratic Risk

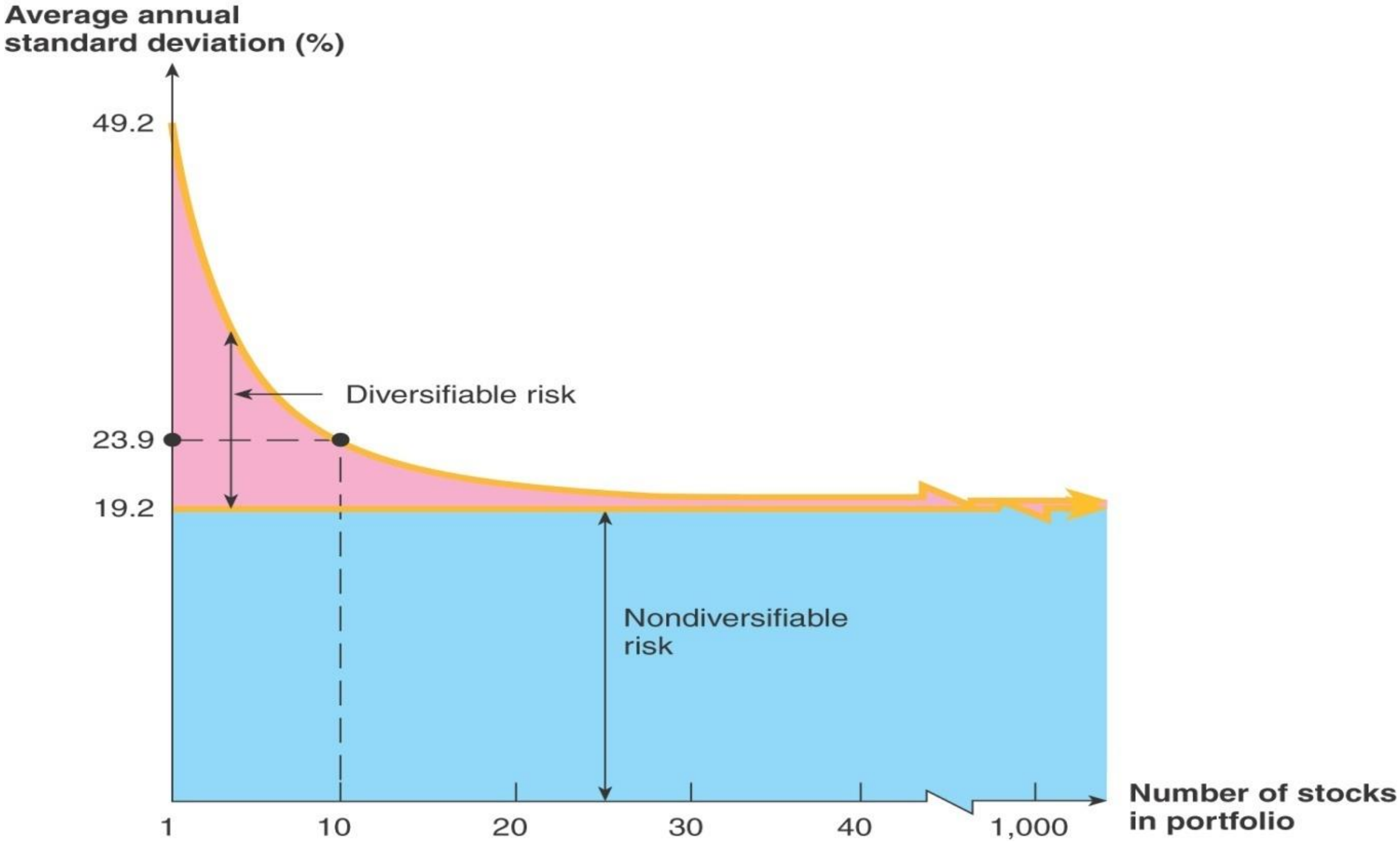
- When held in a portfolio *some* of the risk of a stock disappears
- The risk a stock contributes to the portfolio is LESS than the risk of the stock if held in isolation

$$\begin{pmatrix}\text{total risk of} \\ \text{a stock}\end{pmatrix} = \begin{pmatrix}\text{systematic} \\ \text{risk}\end{pmatrix} + \begin{pmatrix}\text{idiosyncratic} \\ \text{risk}\end{pmatrix}$$

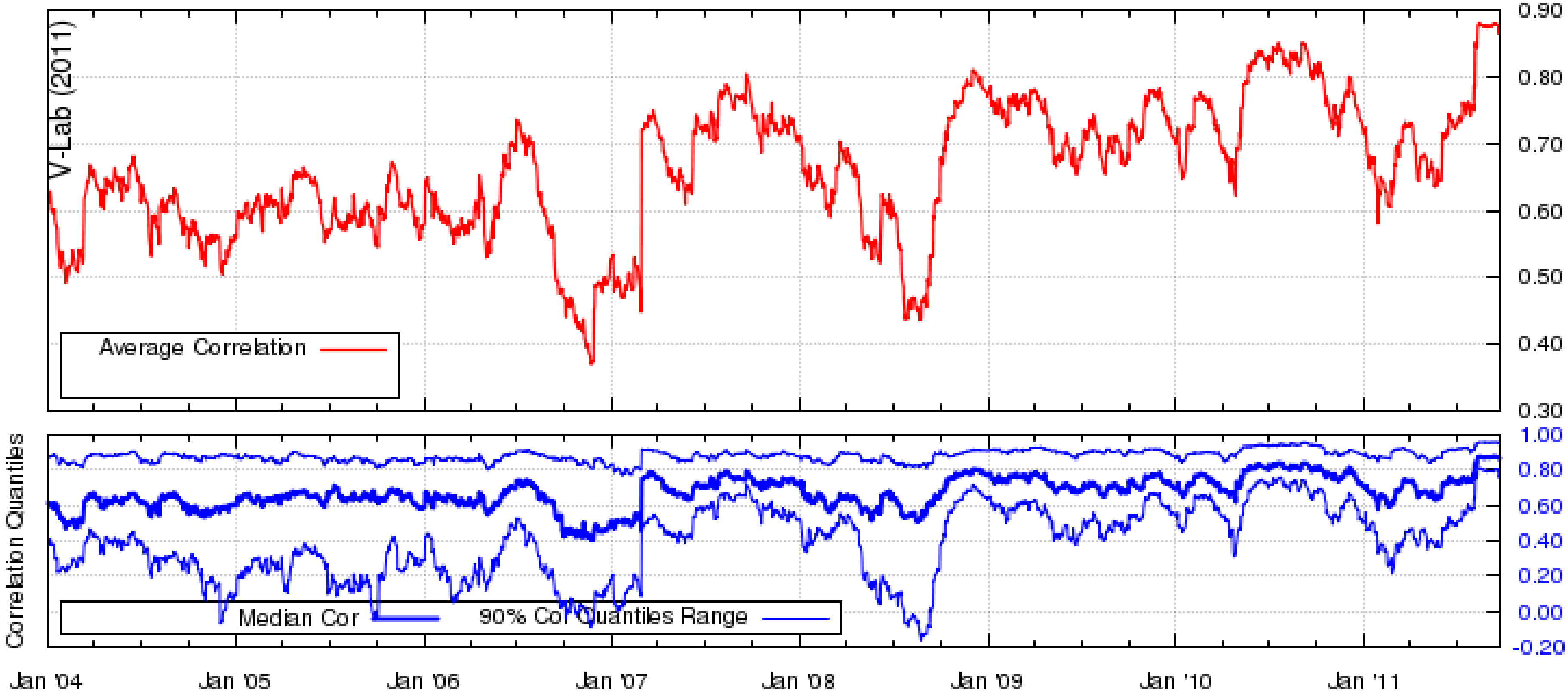
- Investors need to be compensated for holding which kind of risk?

Risk Reduction in Practice

How much reduction in risk should we expect from adding stocks to our portfolio?



Diversification in a Crisis?



Correlations between ~9 US industry sectors

http://vlab.stern.nyu.edu

Implementation Issues

- With *N* stocks, one needs:
  - *N* estimates of expected returns
  - *N* estimates of variances
  - $N(N\text{-}1)/2$ estimates of correlations between all pairs of returns
  - Not every set of estimates is internally consistent
- For N=500, this amounts to 125,750 parameters that need to be estimated

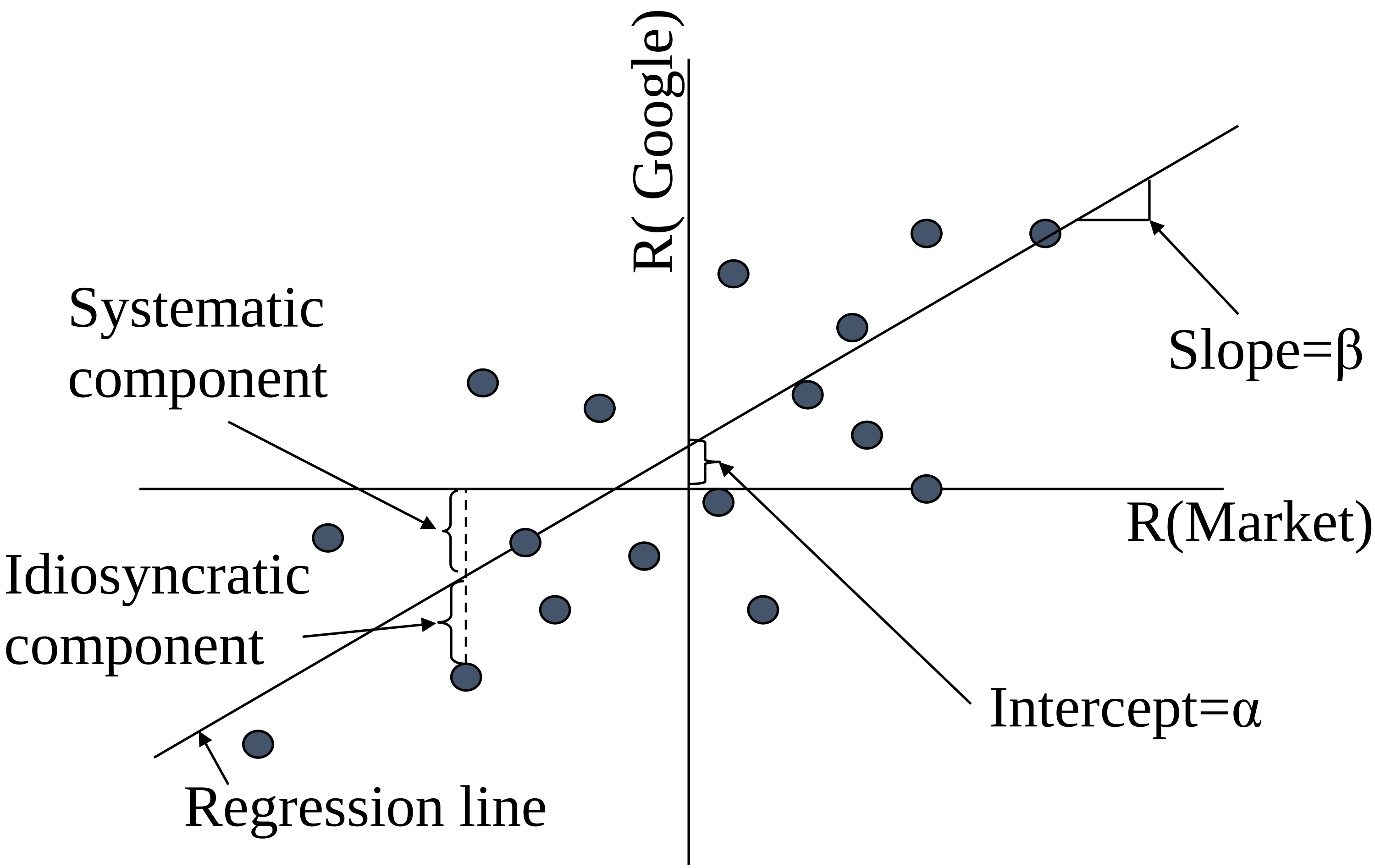
Solutions

1. *Index models*
   All co-movements of returns are captured by a few common factors
2. *The CAPM*
   Implied expected returns in equilibrium

The Single Index Model

- Separating idiosyncratic from systematic risk
- Use excess returns: $R_i = r_i - r_f$
- Fund return: $R_M = \Sigma_i w_i R_i$
- Regression analysis: $R_i = \alpha_i + \beta_i R_M + e_i$
  - e_i: Idiosyncratic component of the return,
    idiosyncratic risk = σ^2_e
  - $\beta_i R_M$: Systematic component of the return,
    systematic risk (covariance risk) = $(\beta_i)^2 \sigma^2_M$
    $\beta_i = \mathrm{cov}(R_M, R_i) / \sigma^2_M$

Estimation



R(Google)
R(Market)
Systematic component
Idiosyncratic component
Slope=β
Intercept=α
Regression line

Conclusion

- Only systematic risk will be priced
- Question: can we “formalize” the notion of systematic vs. idiosyncratic risk, i.e., what is the tangency portfolio?

Assignments

- Reading
  - BKM: Chapters 7.1-7.2
  - Problems: 7.1, 7.4-7.7, 7.9-7.15, 7.17, 7.19, 7.21, CFA 7.1
- Assignments
  - Problem Set 2 due on October 1st