Session 24: Options III

Fall 2025

Calendar

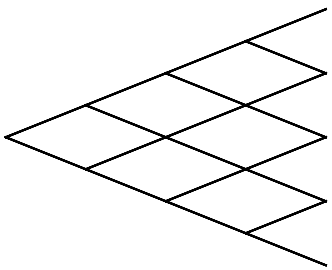
Monday	Wednesday
	19th November – Options III
24th November – Futures	26th November – Swaps
1st – December Problem set #6	3rd December – Review
	10th December – Final Exam

Outline

- Black-Scholes call option pricing
    - Assumptions
    - The formula
    - Intuition
    - Implied volatility
    - Hedge ratio
- Put options

Black-Scholes Valuation

In the binomial tree, allow the stock price to move up or down more frequently, i.e.,



At each point in time, the call option can still be replicated with a portfolio of stock and bonds (dynamic replication). The value of the option is the value of the initial position. As the number of time steps goes to infinity, you get the Black-Scholes formula.

Black-Scholes Assumptions

1. The risk-free interest rate is
   - Constant
   - Continuously compounded
2. The stock price
   - Is log-normal
   - Has constant volatility
3. The stock and risk-free security can be traded continuously at no cost

Black-Scholes Formula

The price of a European call option (on a non-dividend paying stock):

$$C_0 = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / X) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Intuition

- $N(d_1)$ is intuitively related to the probability of the option finishing in-the-money
- If the option is almost certain to be in-the-money at maturity, then
  - $N(d_1) \cong N(d_2) \cong 1$
  - The option price is adjusted intrinsic value: $S_0 - X e^{-rT}$
  - Under what circumstances does this happen?
- If the option is almost certain to be out-of-the-money at maturity, then
  - $N(d_1) \cong N(d_2) \cong 0$, and
  - The option price is close to 0

Intuition cont'd

- What is d_1?

$$\ln(X) \sim N(\mu, \sigma^2)$$

$$E[X] = e^{\mu + \sigma^2/2}$$

$$d_1 = \frac{\ln(S_0 / X) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{S_0 e^{(r+\sigma^2/2)T}}{X}\right)}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{E^*[S_T]}{X}\right)}{\sigma\sqrt{T}}$$

- What is $N(d_1)$?
  The "risk-neutral" probability that $S_T > X$

Determinants of Option Values

	Call	Put
Stock price (today), S_0	___	___
Exercise price, X	___	___
Volatility of stock, σ	___	___
Time to expiration, T	___	___
Interest rate, r	___	___

An Example

S = 105, X = 100, r = 5%, T = 1, σ = 50%

- Intrinsic value: S – X = 5
- Adjusted intrinsic value: S – Xe^{-rT} = 9.88
- Call value:

$$d_1 = 0.4476 \quad N(d_1) = 0.6728$$

$$d_2 = -0.0524 \quad N(d_2) = 0.4791$$

$$C = 25.07$$

Implied Volatility

- For every level of volatility, σ, there is a corresponding option price, C_0
- Similarly, for any option price, C_0, there is a corresponding volatility, σ
- This is called the "implied volatility"
- According to the Black-Scholes model, what should be true about the implied volatility of all options on the same stock?

VIX: Implied Volatility S&P500

http://finance.yahoo.com/

Hedge Ratio (Delta)

- The number of shares of stock in the replicating portfolio is called the hedge ratio or the delta

$$\Delta = \frac{\partial C_0}{\partial S_0} = N(d_1) > 0$$

- If an investment bank writes an option to a client, the bank will hedge its position by buying Δ shares.
- Since Δ is changing over time, the bank must keep adjusting the number of shares held. This is called dynamic hedging.

Put Options

- Recall put-call parity
- Using put-call parity, the Black-Scholes price of a European put option is:

$$
\begin{aligned}
P_0 &= C_0 - S_0 + Xe^{-rT} \\
&= S_0 N(d_1) - Xe^{-rT} N(d_2) - S_0 + Xe^{-rT} \\
&= Xe^{-rT}(1 - N(d_2)) - S_0(1 - N(d_1))
\end{aligned}
$$

- Hedge ratio:

$$\Delta = \frac{\partial P_0}{\partial S_0} = -(1 - N(d_1)) < 0$$

Black Monday (10/19/1987)

- The Black Monday decline was the largest one-day percentage decline in stock market history. The Dow Jones Industrial Average (DJIA) dropped 22.6%.
- Before the crash, many program traders synthetically replicated put options on the index by buying (selling) when the price rose (fell), using the Black-Scholes formula (delta hedging).

Conclusion

Black-Scholes

- Strong assumptions → “clean” result
- Dynamic replication
- The value is the cost of the initial replicating portfolio

Assignments

- Reading
  - BKM: Chapters 17.1-17.5
  - Problems: 17.1-17.3, 17.6-17.7, 17.10-17.11, 17.16, 17.24
- Assignments
  - Problem Set 6 due next week (1st December)