Session 5: Portfolio Theory I

Spring 2025

Outline

- Statistics review
- Arithmetic vs. geometric averages
- Risk and return

Expected Value

- The expected value is the average outcome if the event was repeated infinitely often
- It is the probability-weighted average of the possible outcomes
- Suppose the return r_i on an asset i is equal to $r_i(s)$ with probability $p(s)$ for scenarios/events $s=1,\ldots,S$

$$E[r_i] = \sum_{s=1}^{S} p(s) r_i(s)$$

Expected Return: Walmart

Assume that over the next year:

State	Probability	Return
Recession	25%	-6%
Stagnation	50%	10%
Expansion	25%	22%

$$E[r] = 0.25(-6\%) + 0.50(10\%) + 0.25(22\%) = 9\%$$

Variance and Standard Deviation

- The variance measures how much a variable fluctuates around its mean
- The *variance* is the average squared deviation from the expected value:

$$Var[r_i] = \sigma_i^2 = E\left[(r_i(s) - E[r_i])^2\right]$$
$$= \sum_{s=1}^{S} p(s)(r_i(s) - E[r_i])^2$$

- The *standard deviation* (SD), also called volatility, is the square root of the variance:

$$\sigma_i = \sqrt{\sigma_i^2}$$

Standard Deviation: WMT

State	Probability	Return
Recession	25%	-6%
Stagnation	50%	10%
Expansion	25%	22%

$$E[r] = 9\%$$

$$\begin{aligned} \text{var}[r] &= 0.25(-6\% - 9\%)^2 + 0.50(10\% - 9\%)^2 + 0.25(22\% - 9\%)^2 \\ &= 0.0099 \end{aligned}$$

$$\sigma[r] = \sqrt{0.0099} = 9.95\%$$

Interpretation of the Standard Deviation

How big is a vol. of 9.95%?

Imagine returns were normally distributed (the famous Bell curve):

- 2/3 of a chance that return will be within 1 std dev of mean, in this case [-1%,19%]
- 95% chance within 2 std dev, in this case [-11%,29%]

Covariance

When we start forming portfolios of multiple assets, the risk of the portfolio will depend on the risk of the assets and how much they move together.

Why?

If assets don't always move together one might be up when the other is down, cancelling out the risk.

Covariance

- The *covariance* is the average of the products of the deviations of two variables from their means:

$$\text{cov}[r_i, r_j] = E\left[(r_i - E[r_i])(r_j - E[r_j])\right]$$
$$= \sum_{s=1}^{S} p(s)(r_i(s) - E[r_i])(r_j(s) - E[r_j])$$

- The covariance is:
  - **Positive** if the variables tend to be high and low at the same time
  - **Negative** if the one variable tends to be high when the other is low

Correlation

- Intuitively, the *correlation* measures the same thing as the covariance
- It is defined as the covariance between two variables divided by the product of their standard deviations:

$$\text{corr}[r_i, r_j] = \rho_{ij} = \frac{\text{cov}[r_i, r_j]}{\sigma_i \sigma_j}$$

- The correlation is always between -1 and +1

$$-1 \leq \rho_{ij} \leq 1$$

Correlation: WMT and Kmart

State	Probability	WMT Return	Kmart Return
Recession	25%	-6%	-10%
Stagnation	50%	10%	10%
Expansion	25%	22%	34%

$$E[r_{Km}] = 11\%$$

$$\sigma[r_{Km}] = 15.59\%$$

$$\begin{aligned}\text{cov}[r_{Km}, r_{WMT}] &= 0.25(-10\% - 11\%)(-6\% - 9\%) \\ &+ 0.50(10\% - 11\%)(10\% - 9\%) \\ &+ 0.25(34\% - 11\%)(22\% - 9\%) = 0.0153\end{aligned}$$

$$\rho = 0.0153/[(15.59\%)(9.95\%)] = 0.9784$$

Some Important Results

$$E[r_1 + r_2] = E[r_1] + E[r_2]$$

$$E[w_1 r_1] = w_1 E[r_1]$$

$$\text{var}[r_1 + r_2] = \text{var}[r_1] + \text{var}[r_2] + 2\,\text{cov}[r_1, r_2]$$

$$\text{var}[w_1 r_1] = w_1^2 \,\text{var}[r_1]$$

$$\text{cov}[w_1 r_1, w_2 r_2] = w_1 w_2 \,\text{cov}[r_1, r_2]$$

$$\text{cov}[r_1, r_1] = \text{var}[r_1]$$

$$\text{cov}[r_1 + r_2, r_3 + r_4] = \text{cov}[r_1, r_3] + \text{cov}[r_1, r_4] + \text{cov}[r_2, r_3] + \text{cov}[r_2, r_4]$$

Historical Analogues

Main difficulty getting means, variances and covariances is to know in advance possible states/probabilities/returns.

Logic:

- What we care about is future returns.
- The past is a guide to the future.
- Assume the observed frequency of past events is the frequency going forward, i.e., there is a 1% chance that each of the last 100 years will repeat itself next year.

Historical Analogues

$$\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

$$\mathrm{var}[r] = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2$$

$$\mathrm{cov}[r_i, r_j] = \frac{1}{T}\sum_{t=1}^{T}[r_{it} - \bar{r}_i][r_{jt} - \bar{r}_j]$$

Arithmetic Average Return

Definition

$$\bar{r}_A = \frac{1}{T}\sum_{t=1}^{T} r_t$$

- Useful for forecasting the return next period
- Not equivalent to the holding period return

Geometric Average Return

Definition:

$$[(1+r_1)(1+r_2)(1+r_3)\ldots(1+r_T)]^{1/T}-1$$

$$=\left(\frac{V_T}{V_0}\right)^{1/T}-1$$

- Gives the equivalent per-period return
- Annual HPR

Example: Arithmetic vs. Geometric

- Suppose for an emerging markets fund the return in year 1 is 100% and in year 2 is -50%
- We forecast next year's return to be (100%-50%)/2=25% (arithmetic average)
- Annualized HPR
  - ✓ \$100 invested grows to
    $100(1+1.00)(1-0.50) = \$100$
    ann. HPR = $(100/100)^{1/2}-1=0\%$
  - ✓ Geometric average: $[(1+1.00)(1-0.50)]^{1/2}-1 = 0\%$

Historical Returns

Excess returns, 1926-2009

	Small Stocks	Large Stocks	Long-Term T-Bonds
Avg (arithmetic)	13.72%	7.92%	1.99%
Std. dev.	37.75%	20.81%	8.24%
Min.	-55.34%	-46.65%	-13.43%
Max.	152.88%	54.26%	26.07%

Conclusion

- Risk-return tradeoff
- Portfolio risk and return is all about <u>statistics</u>

Assignments

- Reading
  - BKM: Chapter 6.1, 6.2
  - Handout 10 - Zero Risk Porfolio
  - Handout 11 - Gains from Diversification
- Problems: 6.1, 6.2, 6.8, 6.13
- Assignments
  - Problem Set 2 due February 18^{th} (on Session 10)