Session 6: Portfolio Theory II

Spring 2026

Outline

- Portfolio expected return and variance
- Portfolio choice with 2 risky assets
  - Diversification
  - Investment opportunity set
  - Efficient frontier

Portfolios

A portfolio (p) is

- A combination of N assets, with returns $r_1, \ldots, r_N$
- And portfolio weights $w_1, \ldots, w_N$
  - w_i is percentage of wealth invested in asset i

$$w_i = \frac{\$ \text{ value of position in asset } i}{\text{total } \$ \text{ value of portfolio}}$$

- Portfolio weights sum to one: $w_1 + \ldots + w_N = 1$
- A negative weight indicates a short position

Portfolio Returns

- The return on a portfolio is

$$r_p = \sum_{i=1}^{N} w_i r_i = w_1 r_1 + \ldots + w_N r_N$$

- The **expected** return on a portfolio is

$$E[r_p] = \sum_{i=1}^{N} w_i E[r_i]$$

Portfolio Variance

- With 2 assets (N=2), the portfolio variance is

$$\text{var}[r_p] = \sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

- With N assets, the portfolio variance is

$$\text{var}[r_p] = \sigma_p^2 = \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j>i}^{N} w_i w_j \rho_{ij}\sigma_i\sigma_j$$

$$\sigma_p = \sqrt{\sigma_p^2}$$

Diversification with 2 Risky Assets

- Suppose we have two assets, US and Japan

	Mean (E[r])	Volatility (σ)
US	13.6%	15.4%
Japan	15.0%	23.0%

  with correlation $\rho = 27\% = 0.27$

- If an investor holds $w_1 = 60\%$ in the US and $w_2 = 40\%$ in Japan what are the mean and volatility of the portfolio return?

Gains from Diversification

- Portfolio mean:

  $E[r_p] = 0.6(0.136) + 0.4(0.150) = 14.2\%$
- Portfolio variance:

  $var[r_p] = (0.6)^2(0.154)^2 + (0.4)^2(0.230)^2$

  $+2(0.6)(0.4)(0.27)(0.154)(0.230) = 0.022$

  $\sigma_p = 14.7\%$
- This portfolio has a higher expected return and lower risk than the US market alone!

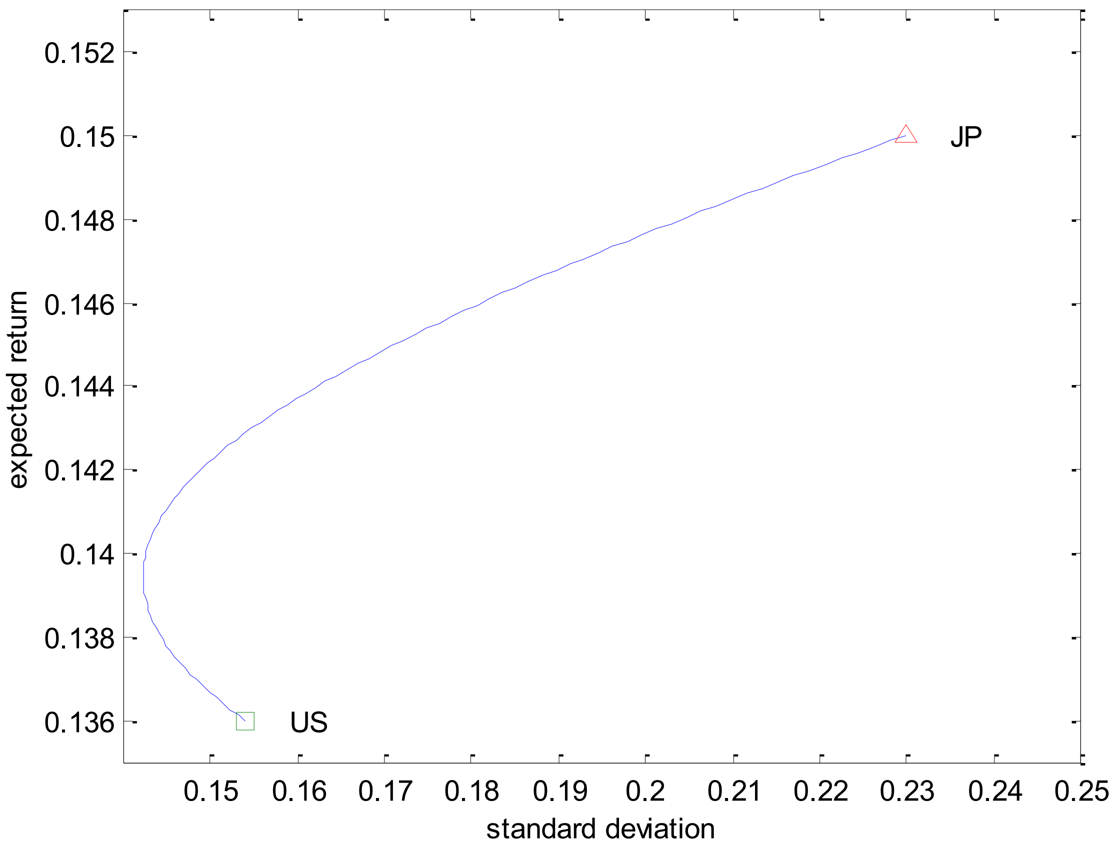
Varying Weights

- Let w be the weight in the US and 1-w the weight in Japan
- The expected return of the portfolio is

  $E[r_p] = w(0.136) + (1-w)(0.150)$
- The variance of the portfolio return is

  $$\text{var}[r_p] = w^2(0.154)^2 + (1-w)^2(0.230)^2 + 2w(1-w)(0.27)(0.154)(0.230)$$
- What happens as we vary the weight w?

The Investment Opportunity Set

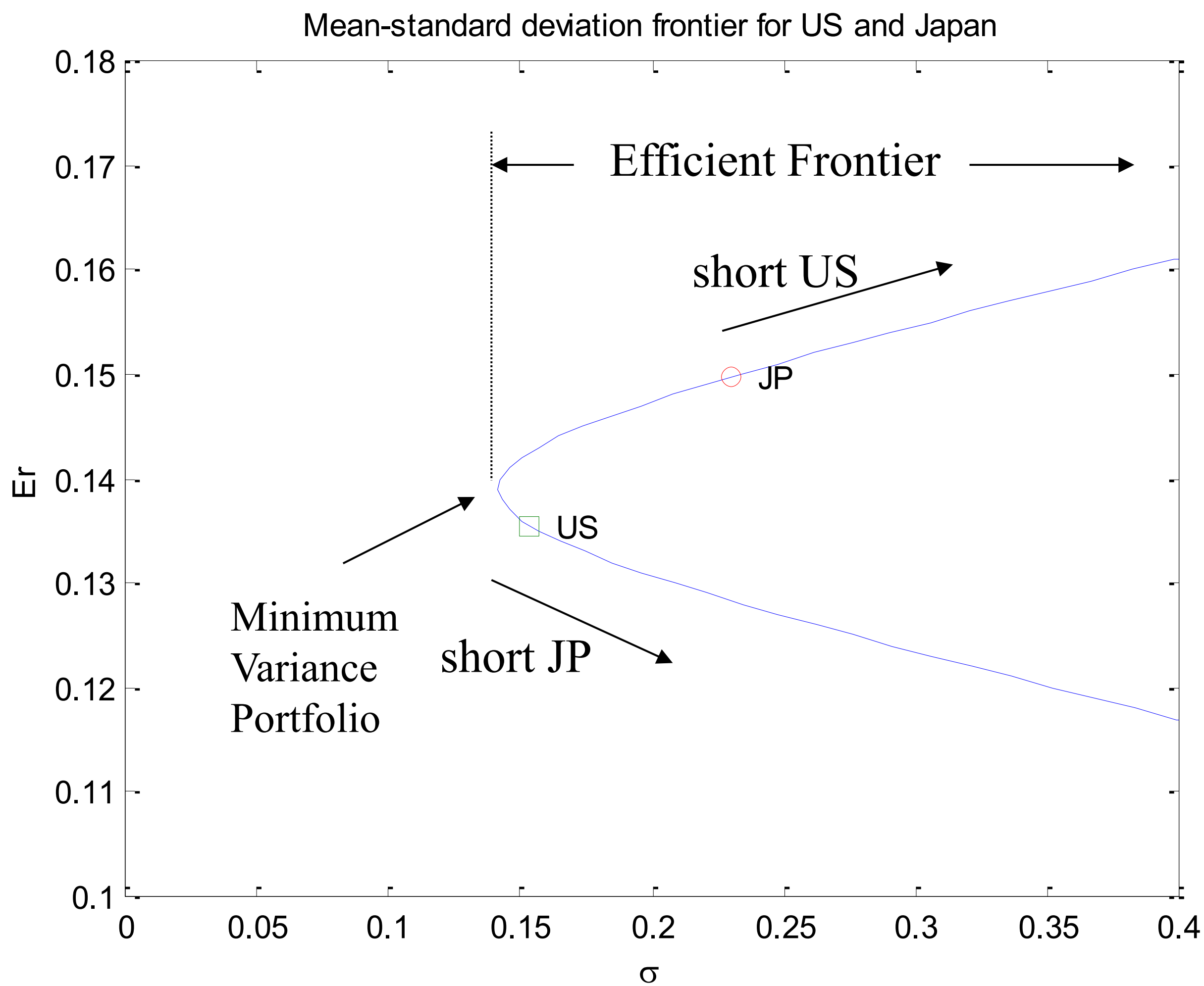


w	mean	volatility
0.0	0.150	0.230
0.1	0.149	0.212
0.2	0.147	0.195
0.3	0.146	0.179
0.4	0.144	0.166
0.5	0.143	0.155
0.6	0.142	0.147
0.7	0.140	0.143
0.8	0.139	0.143
0.9	0.137	0.146
1.0	0.136	0.154

Portfolio Terminology

- The investment opportunity set consists of all available risk-return combinations
- The minimum variance portfolio (mvp) is the portfolio that provides the lowest variance (standard deviation) among all possible portfolios of risky assets
- An efficient portfolio is a portfolio that has the highest possible expected return for a given standard deviation
- The efficient frontier is the set of efficient portfolios. It is the upper portion of the mean-variance (SD) frontier starting at the minimum variance portfolio

The Investment Opportunity Set



Perfect Correlation

$$\mathrm{var}[r_p] = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \rho_{12}\sigma_1\sigma_2$$

- If $\rho_{12} = 1$

$$\mathrm{var}[r_p] = (w_1\sigma_1 + w_2\sigma_2)^2$$

$$\sigma_P = w_1\sigma_1 + w_2\sigma_2$$

→ <u>no</u> gains from diversification

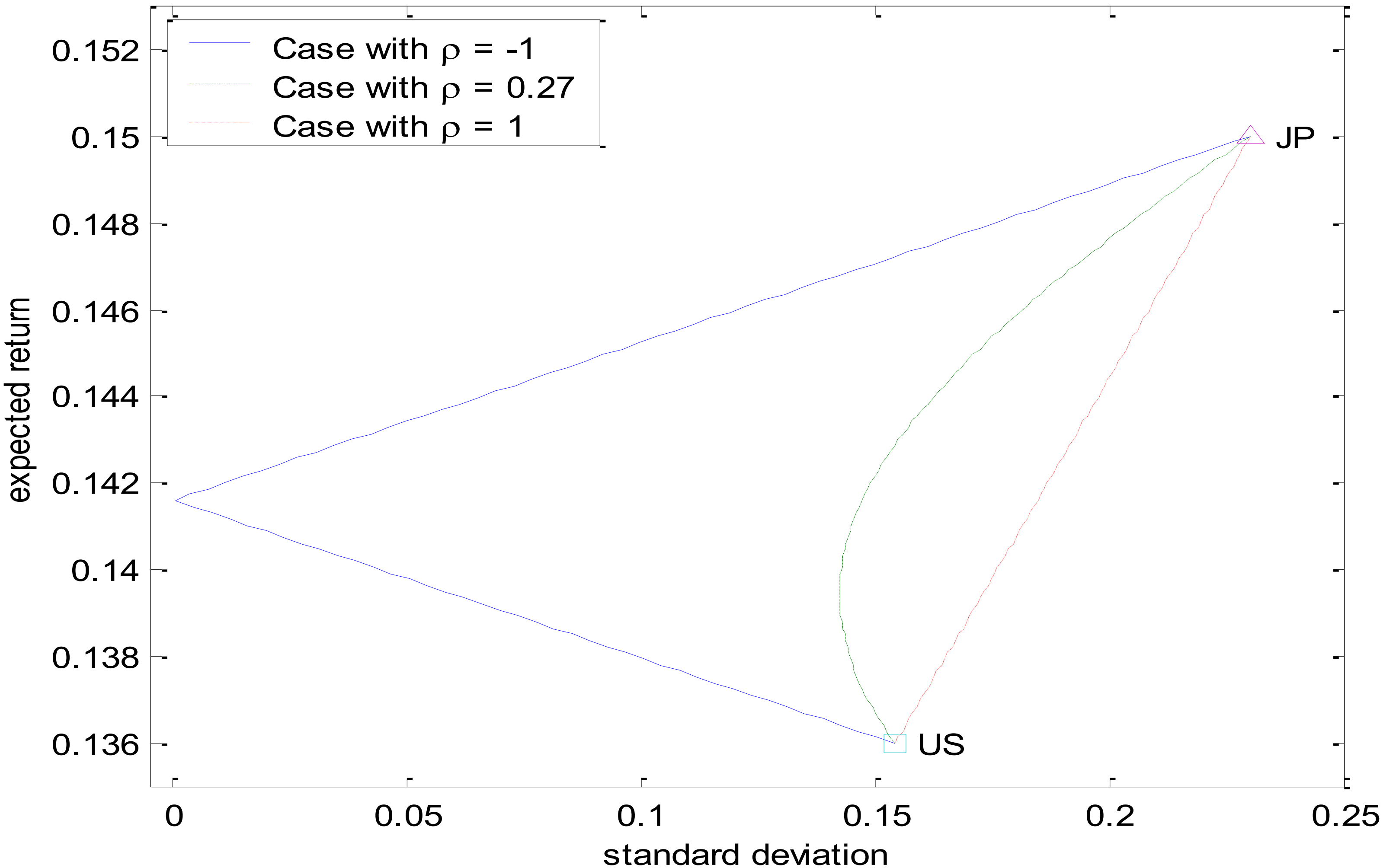
- If $\rho_{12} = -1$

$$\mathrm{var}[r_p] = (w_1\sigma_1 - w_2\sigma_2)^2$$

$$\sigma_P = | w_1\sigma_1 - w_2\sigma_2 |$$

→ <u>maximal</u> gains from diversification

Varying Correlations



A Zero Risk Portfolio

- Assume that the correlation between the US and Japanese stock markets is -1
- Determine the portfolio weights in Japan and the US that create a zero risk portfolio.
  (Hint: $w_1 + w_2 = 1$)

Conclusion

- Diversification—the (only?) finance free lunch
- It’s all about correlation!

Assignments

- Assignments
  - Problem Set 2 due on 18th February