Session 8: Portfolio Theory IV

Spring 2026

Outline

- The risk-free asset
- Portfolio selection with a risk-free asset and 1 risky security
- Portfolio selection with a risk-free asset and 2 risky securities

Asset Allocation

“The fundamental decision of investing is the allocation of your assets: How much should you own in stock? How much should you own in bonds? How much should you own in cash reserves? That decision accounts for an astonishing 94% of the difference in total returns achieved by institutional investors. There is no reason to believe that the same relationship does not hold for individual investors.”

John Bogle, Founder of the Vanguard Group

Properties of the Risk-Free Asset

- The risk-free return is denoted r_f
- The risk-free return is known for sure

$$E[r_f] = r_f$$

$$\mathrm{var}[r_f] = 0$$

$$\mathrm{cov}[r_f, r_i] = 0 \quad \forall i \quad (\rho_{fi} = 0)$$

A Risk-Free and a Risky Asset

- Let w be the fraction of wealth invested in the risky asset i (the rest is invested in the risk-free asset f)
- Expected portfolio return:

$$E[r_p] = w\,E[r_i] + (1-w)r_f = r_f + w\,\underbrace{E[\overbrace{r_i - r_f}^{\textit{excess return}}]}_{\substack{\textit{expected excess return} \\ (\textit{risk premium})}}$$

- Variance of portfolio return

$$\sigma_p^2 = w^2\sigma_i^2 + (1-w)^2\sigma_f^2 + 2w(1-w)\sigma_i\sigma_f\rho_{if} = w^2\sigma_i^2$$

$$\sigma_p = |w|\,\sigma_i \quad \Rightarrow \quad |w| = \frac{\sigma_p}{\sigma_i}$$

Investment Opportunity Set

- Consider various portfolios *p* (which are long in the risky asset and long or short in the risk-free asset)
- What is the risk-return relationship?

$$E[r_p] = r_f + \underbrace{\frac{E[r_i - r_f]}{\sigma_i}}_{\textit{Sharpe Ratio}} \sigma_p$$

Sharpe Ratio =price of risk

$$E[r_p] = r_f + SR_i \sigma_p$$

The Capital Allocation Line

The Capital Allocation Line

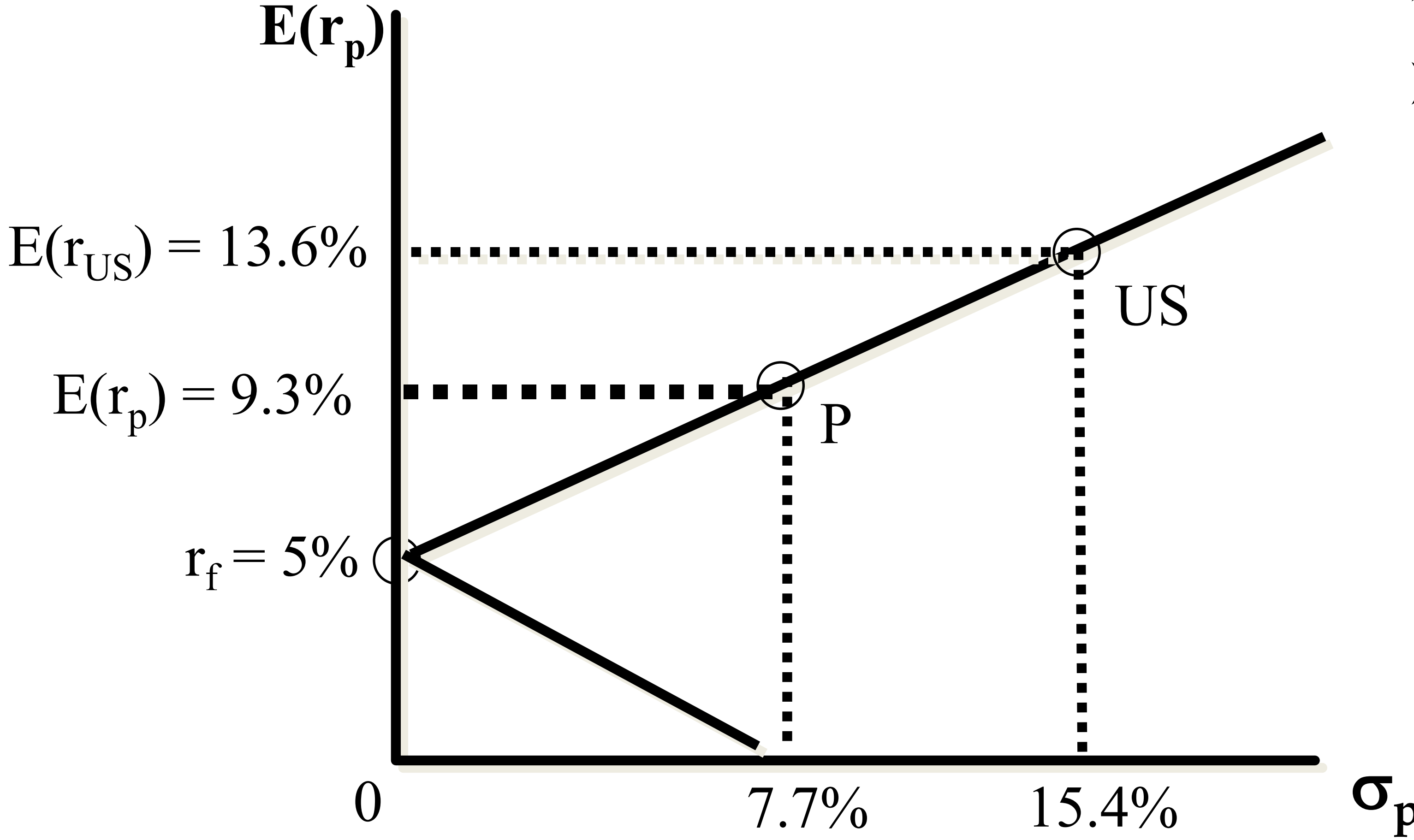
- Risky asset: US stock market
  $E[r_{US}] = 13.6\%$, $\sigma_{US} = 15.4\%$
- Risk-free: US T-bill
  $r_f = 5\%$
- Capital allocation line

$$E[r_p] = r_f + SR_{US}\sigma_p$$

$$\mathrm{SR}_{\mathrm{US}} = \frac{\mathrm{E}[\mathrm{r}_{\mathrm{US}}] - \mathrm{r}_{\mathrm{f}}}{\sigma_{\mathrm{US}}} = \frac{0.136 - 0.05}{0.154} = 0.56$$

The Capital Allocation Line

- $w = 0$
- $w = 1$
- $w = 0.5$
- $w = -0.5$



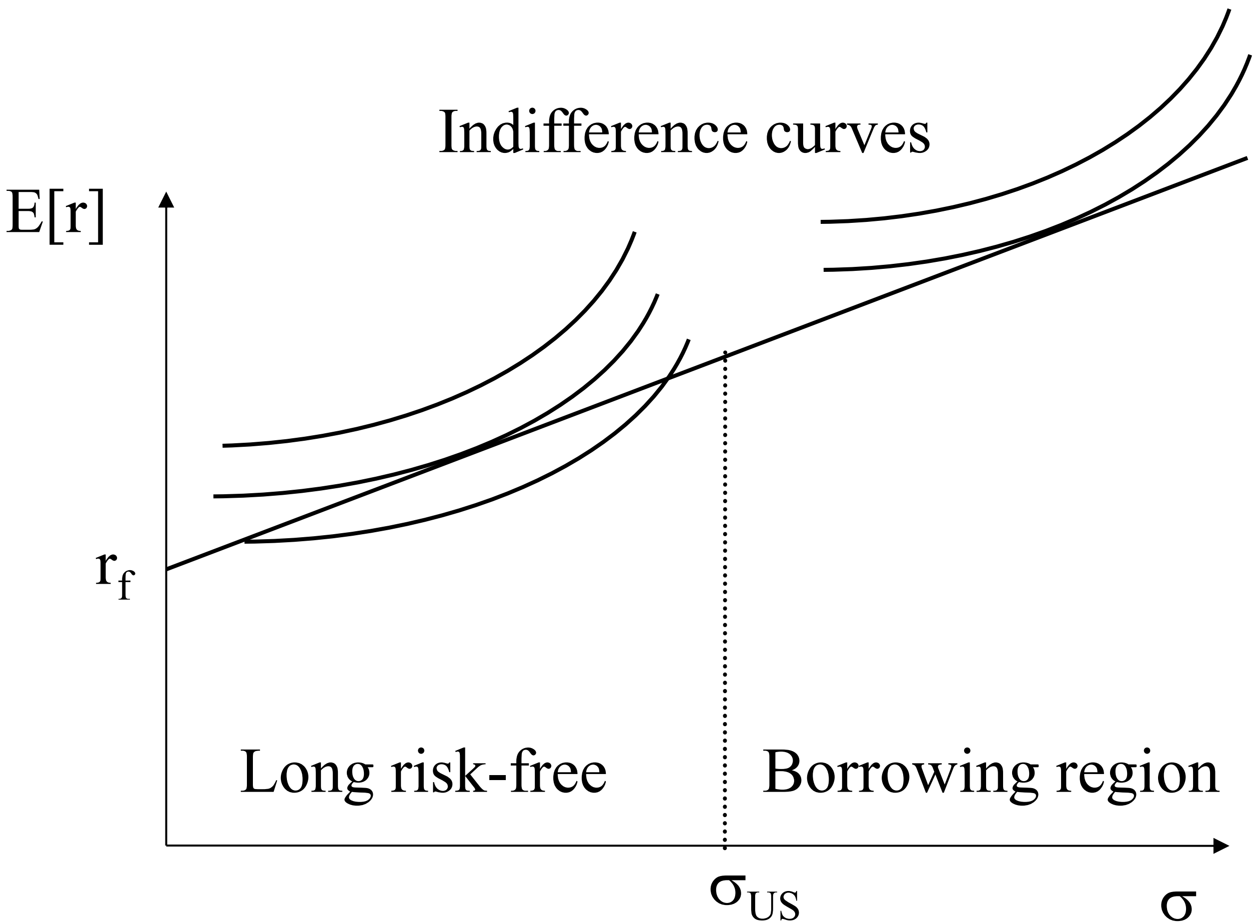
The Capital Allocation Line

- How can I get a 17.9% expected return by investing in US stocks and the risk-free asset?

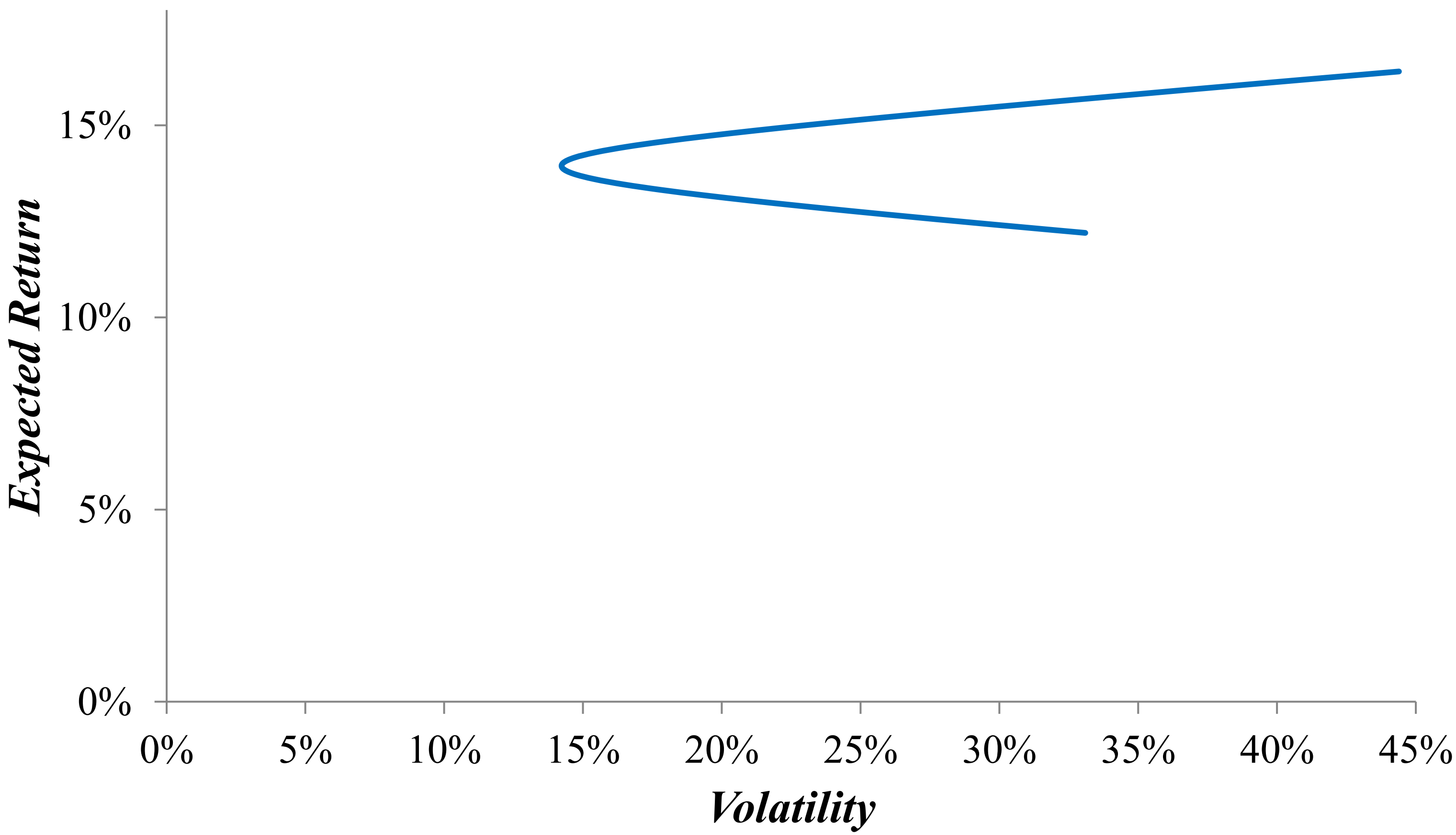
- What is the standard deviation for this portfolio?

- Can you do this in the real world?

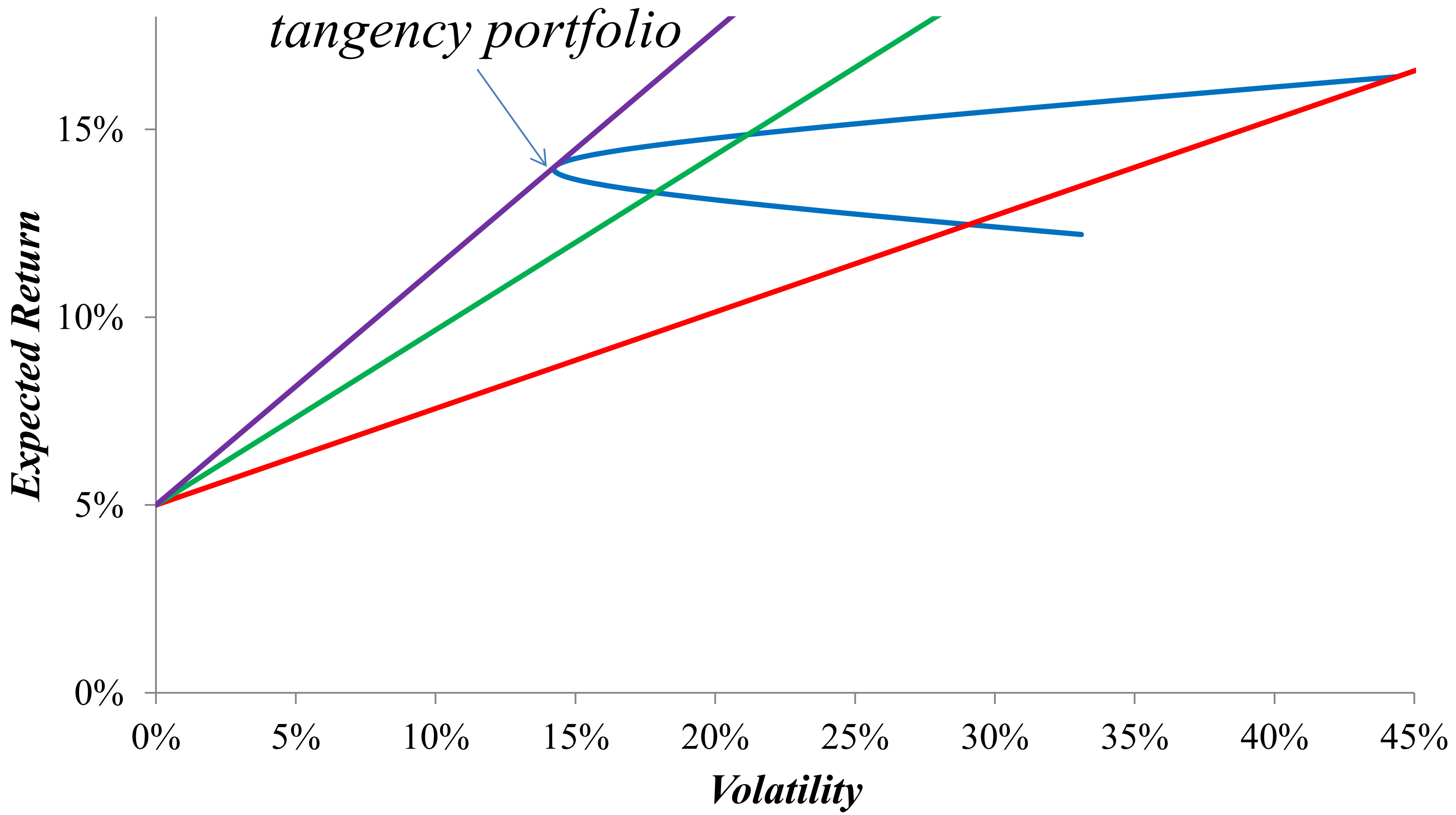
Optimal Portfolio Choice



Two Risky Assets



Two Risky and A Risk-Free Asset



Optimal Portfolio Selection

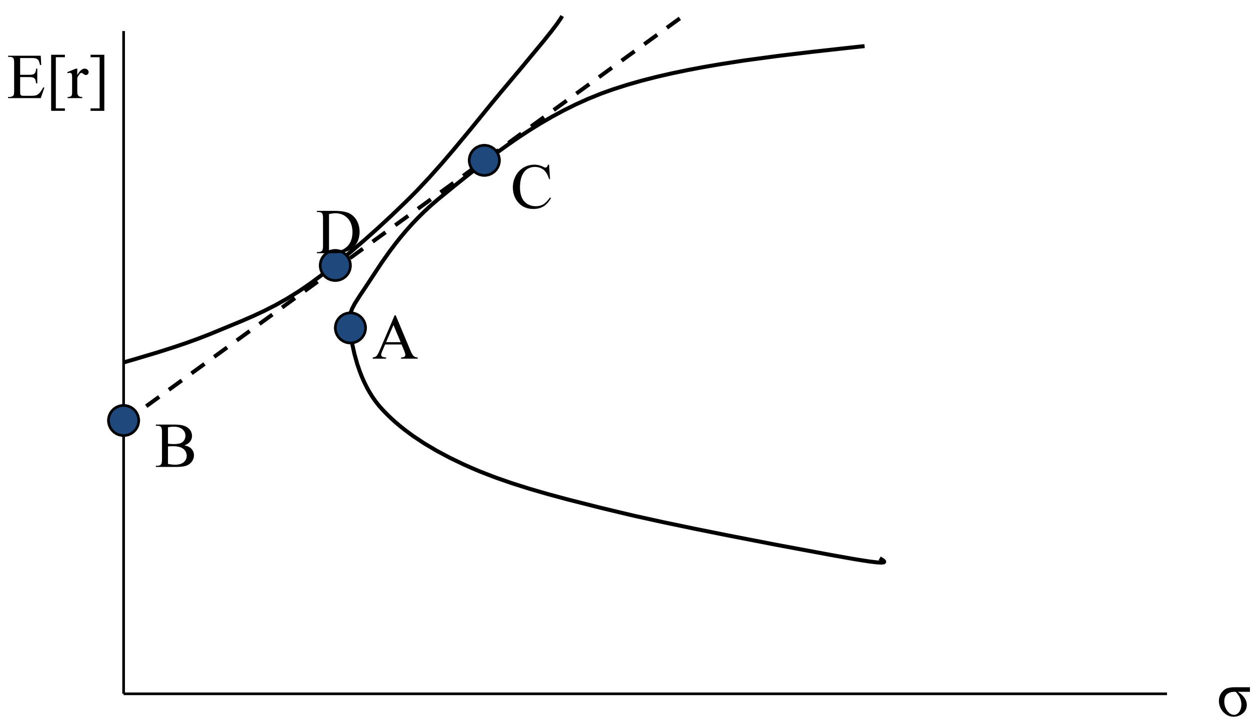
1. Create the set of possible mean-SD combinations from different portfolios of risky assets
2. Find the "tangency portfolio," that is, the portfolio with the highest Sharpe ratio:

$$SR_i = \frac{E[r_i] - r_f}{\sigma_i}$$

3. Choose the combination of the tangency portfolio and the risk-free asset to suit your risk-return preferences

Optimal Portfolio Selection

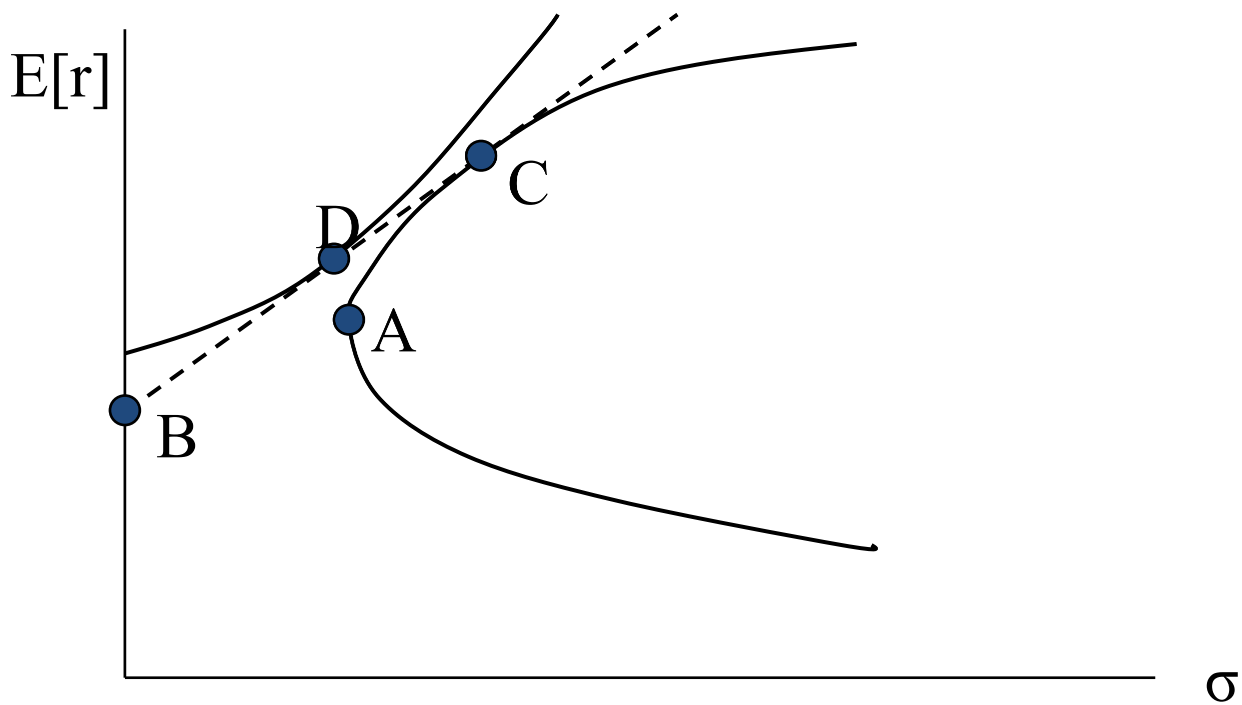
What are the names of the portfolios corresponding to the following points?



Optimal Portfolio Selection

Suppose portfolio C consists of 60% US, 40% JP, and portfolio D has 40% allocation to the risk-free asset.

What is the composition of D?



Two-Fund Separation

- All investors hold combinations of the same two "mutual funds":
  - The risk-free asset
  - The tangency portfolio
- An investor's risk aversion determines the fraction of wealth invested in the risk-free asset
- But, all investors should have the rest of their wealth invested in the tangency portfolio

Assignments

- Reading
  - BKM: Chapters 6.4-6.6
  - Problems: 6.9-6.12, 6.16-6.18, 6.21
- Assignments
  - Problem Set 2 due on February 18^{th}