Foundations of Finance

Time Value of Money

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Partial Review of Last Class

- How do primary markets differ from secondary markets?
- What is the process for selling short?
- What is bid-ask spread? How does a market maker use it to control inventory?

Topics

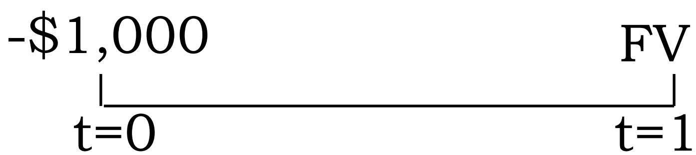
- Time value of money
- Future Value (FV), Present Value (PV) and Yield
- Multiple payment securities
  - Annuities and perpetuities
- Quoted versus effective annual rates

Introduction

- All financial decisions involve a tradeoff between cash outlays *now* and cash inflows *in the future* (or vice versa)
- Should you get a degree? Pay tuition and forego income for 4 years in exchange for higher income in your future career
- Should a company take out a loan? Obtain funds from a bank now, pay back later with interest
- Need to compare cash flows *at different points in time*

Future Value

- What is the future value of $1,000 invested for one year at an annual rate of 5%



$$FV = PV \times (1+r)^T = 1{,}000 \times (1.05)^1 = \$1050$$

Future Value

- What is the value of $1,000 invested for two years at an annual rate of 5%?

$$FV = PV \times (1+r)^T = 1{,}000 \times (1.05)^2 = \$1102.5$$

- The final payment can be decomposed as:

$$1102.5 = \underbrace{\$1000}_{\text{Principal}} + \underbrace{2 \times 0.05 \times \$1000}_{\text{Simple Interest}} + \underbrace{0.05 \times 0.05 \times \$1000}_{\text{Interest on Interest}}$$

Future Value

- $FV = PV \times (1 + r)^T$

- Future Value (FV) measures the nominal future sum of money that a given sum of money today is "worth" at a specified time in the future assuming a certain rate of return.

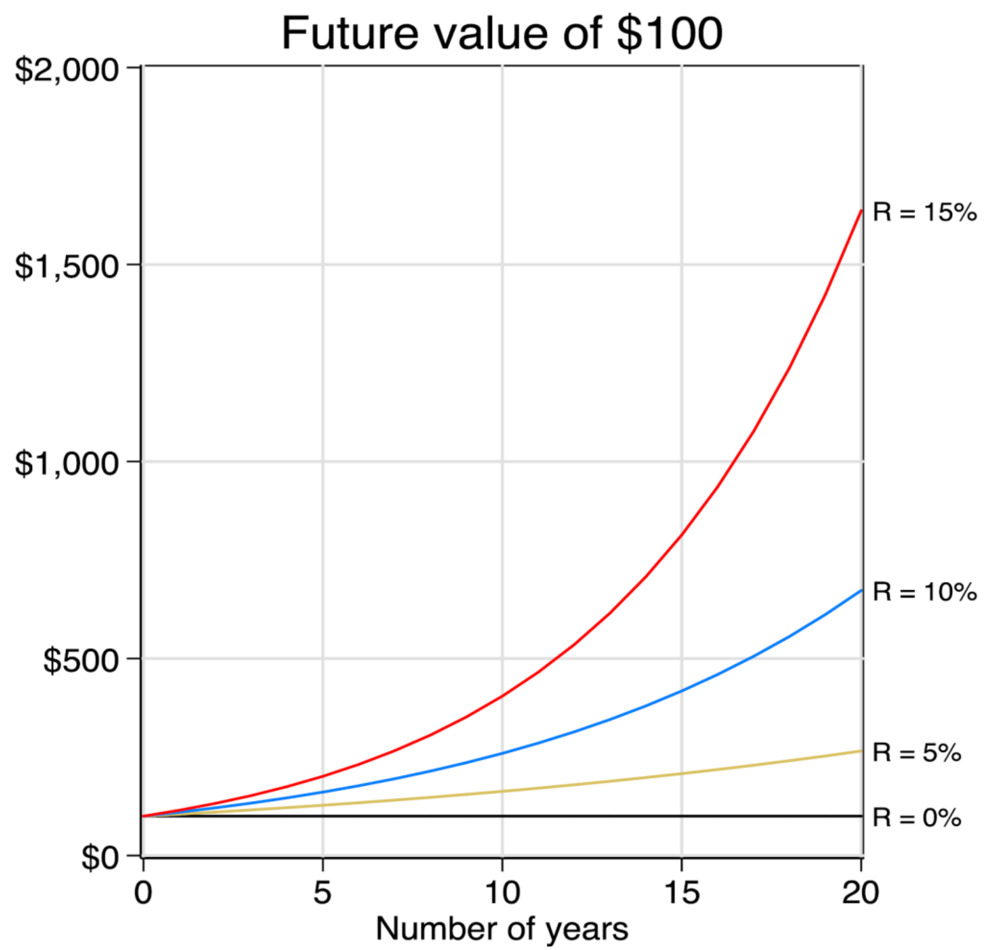
- The Future Value consists of the Present Value (the principal), simple Interest, and interest on interest.

Future Value Formula

The future value of investing an amount PV for T years when the interest rate is R is

$$FV = PV \times (1 + R)^T$$

	0%	5%	10%	15%
0	100	100	100	100
5	100	128	161	201
10	100	163	259	405
15	100	208	418	814
20	100	265	673	1,637



New York Real Estate Bargain

- The Dutch West India Company dispatched the first permanent settlers to Manhattan Island in 1624. They established the town of New Amsterdam as more settlers arrived.

- In 1626, the fledgling town's governor, Peter Minuit, bought Manhattan—meaning "Island of Hills"—from the Canarsie Indians for 24 dollars' worth of beads and trinkets. New Yorkers often call this the last real estate bargain in New York.

- How good was it a bargain if the interest rate was 5%?
- What is the interest rate was 6%?

New York Real Estate Bargain

- Suppose interest rates are 6%, at the beginning of 2026

$$FV = PV \times (1+r)^T = \$24 \times (1.06)^{400} = \$318{,}095{,}369{,}845$$

- Suppose rates are 7%:

$$FV = PV \times (1+r)^T = \$24 \times (1.07)^{400} = \$1.36 \times 10^{13}$$

- A future value of 13.6 trillion!

Present Value (PV)

- How much do you need to invest today, with an interest rate of r, to have $FV next year?

$$PV = \frac{FV}{(1+r)^t}$$

- Discount factor or the Present Value factor is $\frac{1}{(1+r)^t}$
- Present value factor tells us how to convert future value to present value
- We say we calculate PV by *discounting* the FV at a *discount rate* r

Present Value Example

- To receive $1000 in one year, how much should I invest today at a rate of 5%?

$$PV = \frac{FV}{(1+r)^t} = \frac{1000}{(1+5\%)^1} = 952.38$$

- To receive $1000 in two years, how much should I invest today at a rate of 5%?

$$PV = \frac{FV}{(1+r)^t} = \frac{1000}{(1+5\%)^2} = 907.03$$

Yield

- Now let's introduce another term, the yield.
- At what hypothetical constant rate does $PV grow to exactly $FV in a span of T years?

$$r = \left({}^{FV}/_{PV}\right)^{1/T} - 1$$

- For a bond, the yield also called yield-to-maturity.
- For a single payment bond, the interest rate and the yield are the same thing. More complicated for multiple-payment bonds.

Yield

- Microsoft seeks to raise \$1.3 billion from investors in exchange for promising to repay \$2.5 billion on 25 years.
- What yield is Microsoft offering?

Yield

- Microsoft seeks to raise \$1.3 billion from investors in exchange for promising to repay \$2.5 billion on 25 years.
- What yield is Microsoft offering?
- $r = (2.5 / 1.3)^{(1/25)} – 1 = 2.65\%$
- A single payment: the interest rate and the yield are the same thing

Time

- Your aunt is 40 years old and has $500,000 saved for retirement. She wants to retire with $4 million without any additional savings.
- If the rate of return on her savings is 8%, at what age can she retire?

Time

- Your aunt is 40 years old and has $500,000 saved for retirement. She wants to retire with $4 million without any additional savings.
- If the rate of return on her savings is 8%, at what age can she retire?
- $t = \log(4{,}000{,}000 / 500{,}000) / \log(1.08) = 27$ years
- She can retire at 67
- Doubling three times ($0.5 \rightarrow 1 \rightarrow 2 \rightarrow 4$) at 8%: about 9 years each

Recap

- In general

$FV = PV \times (1+r)^t$	$r = (\frac{FV}{PV})^{\frac{1}{t}} - 1$
$PV = \frac{FV}{(1+r)^t}$	$t = \frac{\log(\frac{FV}{PV})}{\log(1+r)}$

- Given any 3, you can find the 4th variable

Practice Questions

- If you invested \$1,000 and received \$1,108 after 3 years, what was the interest rate?

- If you invested \$1,000 at 6% and received \$1,791, for how long did you invest?

- You want to have \$1,000 in four year's time. Interest rates are 2%. How much do you need to invest today?

Practice Questions

- If you invested \$1,000 and received \$1,108 after 3 years, what was the interest rate?
- If you invested \$1,000 at 6% and received \$1,791, for how long did you invest?
- You want to have \$1,000 in four year's time. Interest rates are 2%. How much do you need to invest today?
- $r = (1{,}108/1{,}000)^\wedge(1/3) - 1 = 3.48\%$
- $t = \log(1.791) / \log(1.06) = 10$ years
- $PV = 1{,}000 / 1.02^4 = \$923.85$

Multiple Cash Flows

- The present value of a set of cash flows CF_t received in $t = 1, 2, \dots, T$ years is equal to the sum of the present values of the individual cash flows:

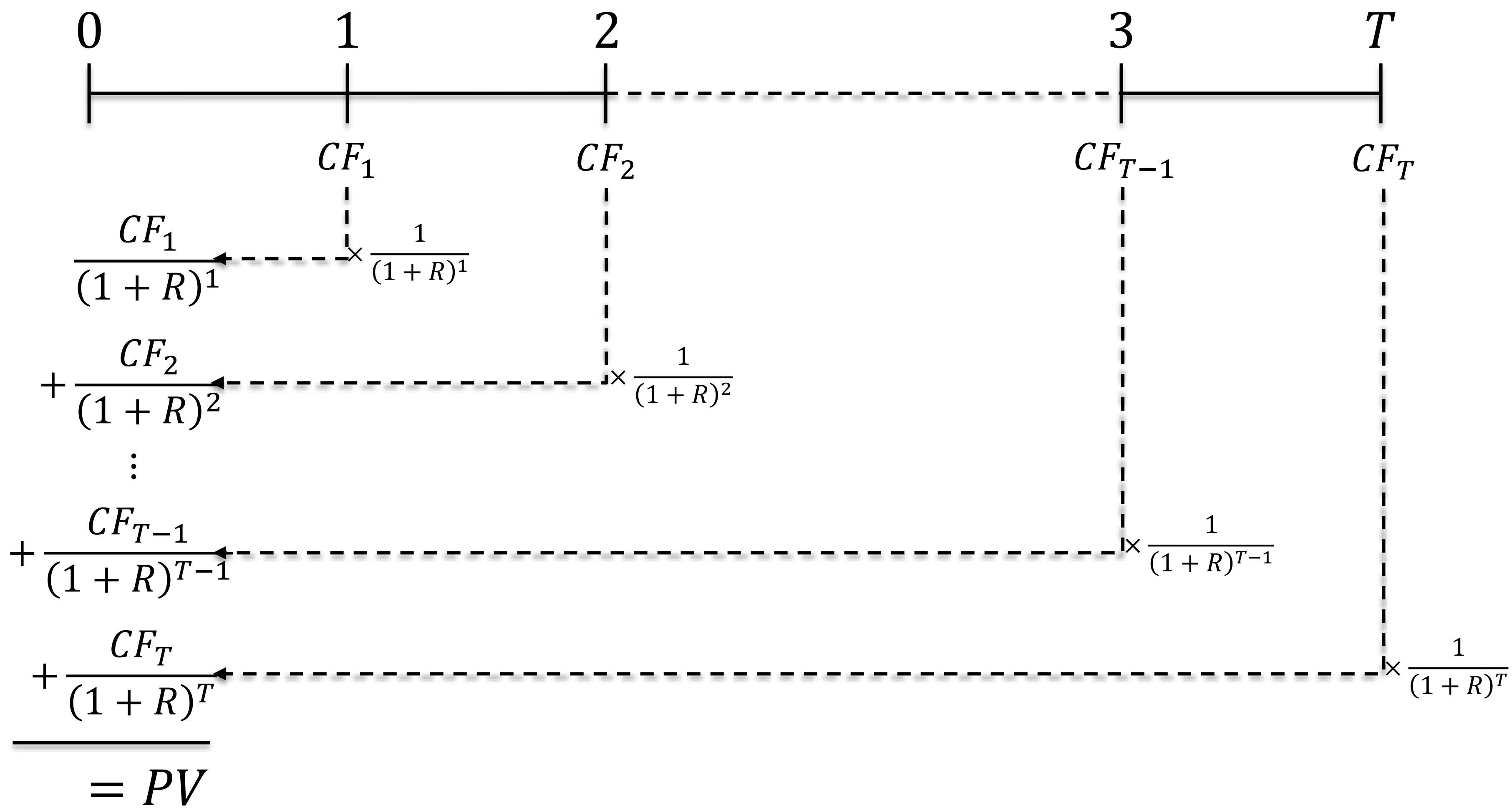
$$PV = \frac{CF_1}{(1+r)^1} + \frac{CF_2}{(1+r)^2} + \cdots + \frac{CF_T}{(1+r)^T} = \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t}$$

- Similarly, the future value of a set of cash flows CF_t is:

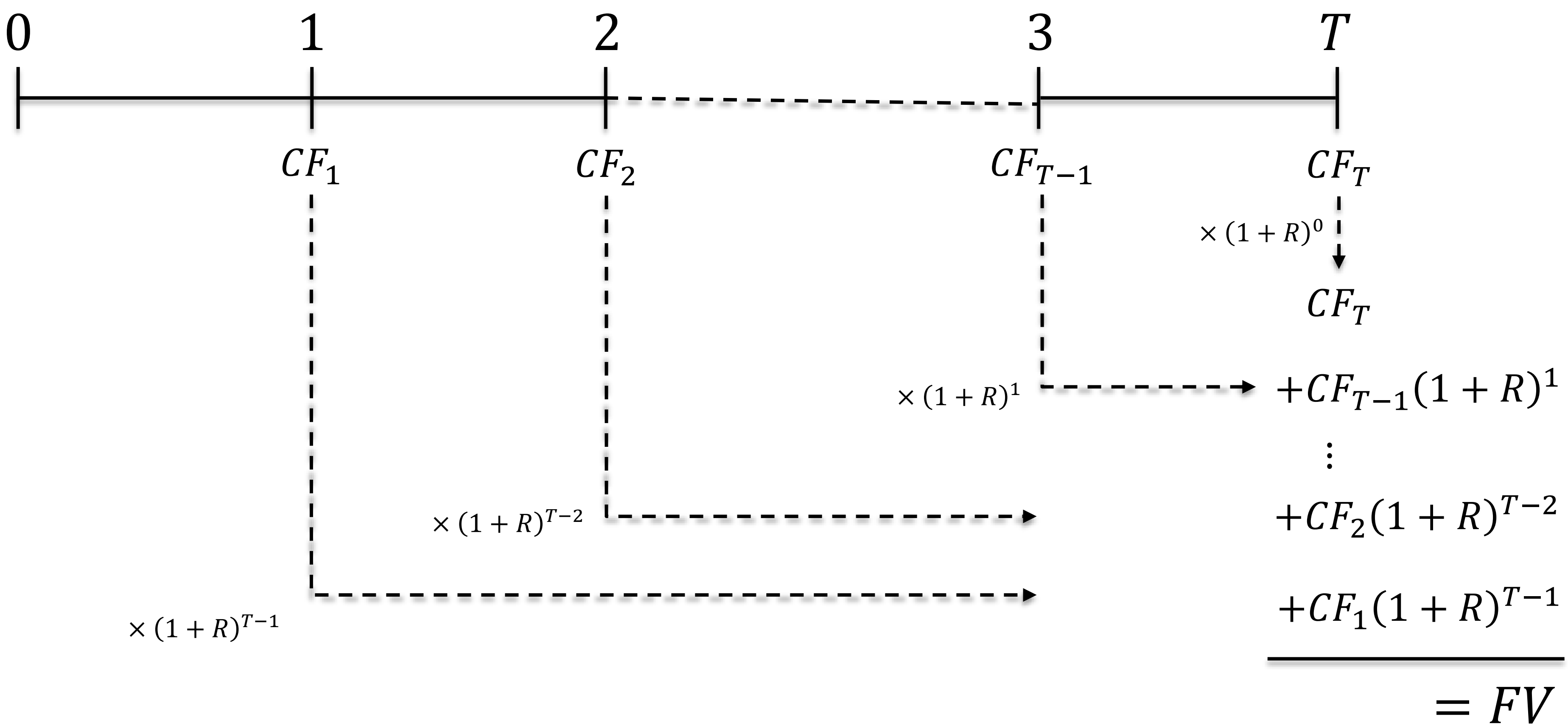
$$FV = CF_1(1+r)^{T-1} + CF_2(1+r)^{T-2} + \cdots + CF_T$$
$$= \sum_{t=1}^{T} CF_t(1+r)^{T-t}$$

- Note that, as before, $\mathrm{FV} = PV(1+r)^T$

Multiple Cash Flows - PV



Multiple Cash Flows - FV



PV and FV of Multiple Cash Flows

- Your company is considering buying an office building that it plans to rent out for $40 thousand per year (net of expenses) for 5 years, then sell for $500 thousand
- How much should your company willing to pay for the building if the appropriate discount rate is 6%?
- How much cash will your company have from collecting rent and selling the building after 5 years? Assume your company earns 6% interest on its cash

PV and FV of Multiple Cash Flows

- Buy an office building: rent \$40k a year for 5 years, then sell for \$500k. Discount rate 6%.
- PV = $\Sigma\, 40/1.06^t + 500/1.06^5$ = \$542.1k

- FV after 5 years = $\Sigma\, 40 \cdot 1.06^{(5-t)} + 500$ = \$725.5k
- Check: $542.1 \times 1.06^5 = 725.5$ — one value, two dates

Perpetuities

- Let's look at a couple of special fixed-income assets.
- A perpetuity or a consol is a bond that pays a fixed payment C forever. No principal payment. By convention, the payment in the current period has already happened.
- PV: Draw the timeline of payments. Note that our definition assumes the payment for t=0 has already happened.

$$
\begin{array}{cccccccc}
0 & C & C & C & C & C & C & \cdots \\
t=0 & 1 & 2 & 3 & 4 & 5 & 6 &
\end{array}
$$

- Calculate the present value of each of these payments:

$$
\begin{array}{cccccccc}
0 & \frac{C}{(1+r)} & \frac{C}{(1+r)^2} & \frac{C}{(1+r)^3} & \frac{C}{(1+r)^4} & \frac{C}{(1+r)^5} & \frac{C}{(1+r)^6} & \cdots \\
t=0 & 1 & 2 & 3 & 4 & 5 & 6 &
\end{array}
$$

Perpetuity Timeline

- Therefore, the present value of the entire stream of payments is:

$$PV = \sum_{i=1}^{\infty} \frac{C}{(1+r)^i}$$

$$PV = \sum_{i=1}^{\infty} \frac{C}{(1+r)^i} = C \times \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} = \frac{C}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i$$

- By formula for geometric sums

$$PV = \frac{C}{1+r} \times \frac{1}{1 - \frac{1}{1+r}} = \frac{C}{r}$$

Perpetuity Example

That Debt From 1720? Britain's Payment Is Coming

After that financial crash in 1720, called the South Sea Bubble, the British government was forced to undertake a bailout that eventually left several million pounds of debt on its books. Almost three centuries later, Britons are still paying interest on a small part of that obligation.

George Osborne, the chancellor of the Exchequer, said this month that in 2015 Britain would repay part of the country's debt from World War I, and that he wanted to pay off other bonds for debt incurred in the 18th and 19th centuries.

The debt originating in part from the South Sea Bubble, the oldest still on the books, was consolidated into bonds issued in 1853, and those who now own them receive an annual payout of 2.5 percent.

Perpetuity: Example

- How much is a preferred stock worth if it pays $10 dividend and if the discount rate is 8%?
- If I invest $125 at 8%, I will earn $10 ($= 0.08 \times 125$) interest in a year. If I then re-invest the $125, I will earn another $10 in two years, and so on. The cash flows are the same as the preferred stock, so the preferred stock is worth $125
- The value of a perpetuity satisfies $R \times PV = CF$ or

$$PV = \frac{CF}{R}$$

Constant Growth Perpetuities

- What if the cash flows of the perpetuity grow at a constant rate g?

0	1	2	3	4
	CF	$CF(1+g)$	$CF(1+g)^2$	$CF(1+g)^3$

- Ex: common stock pays $10 dividend starting in year 1 that grows 4% a year. The discount rate is still 8%.
- Now I need to invest $250 so that the interest of $20 (=0.08*250) covers both the $10 dividend and another $10 to grow the principal by 4%
- So, a constant-growth perpetuity satisfies $R \times PV = CF + g \times PV$ or $PV = \frac{CF}{R-g}$

Perpetuity Example

- Consider a perpetuity that pays \$1,000 forever, however, only starting 10 years from now. Interest rates are 5%. What is the PV?

Perpetuity Example

- Consider a perpetuity that pays $1,000 forever, however, only starting 10 years from now. Interest rates are 5%. What is the PV?
- A perpetuity formula values it one period before the first payment
- At $t = 9$: $PV_9 = 1{,}000 / 0.05 = \$20{,}000$
- Today: $PV_0 = 20{,}000 / 1.05^9 = \$12{,}892$

Annuities

- Definition: Pays a fixed cash flow C for T periods (starting at time 1)

$$PV = \sum_{i=1}^{T} \frac{C}{(1+r)^i} = C \sum_{i=1}^{T} \frac{1}{(1+r)^i} = \frac{C}{(1+r)} \sum_{i=0}^{T-1} \left(\frac{1}{1+r}\right)^i$$

- By formula for geometric sums:

$$PV = \frac{C}{1+r} \frac{1-\left(\frac{1}{1+r}\right)^T}{1-\frac{1}{1+r}} = \frac{C}{r} - \frac{C/(1+r)^T}{r} = C\left[\frac{1-\frac{1}{(1+r)^T}}{r}\right]$$

$$PV = \underset{\text{Payment}}{C}\underset{\text{PV factor}}{\left[\frac{1-\frac{1}{(1+r)^T}}{r}\right]}$$

Annuities

- Definition: Pays a fixed cash flow C for T periods (starting at time i)
- Consider the following annuity

$t = 0$	1	2	3	4	5	6	…
0	C	C	C	0	0	0	…

- Also consider perpetuity A

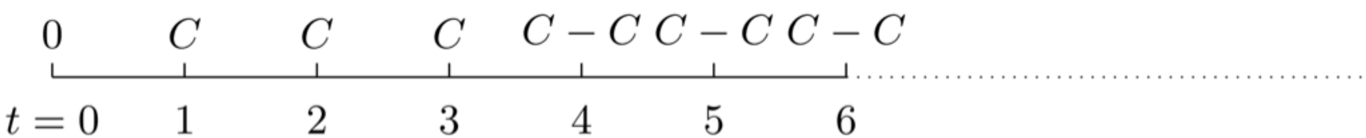
$t = 0$	1	2	3	4	5	6	…
0	C	C	C	C	C	C	…

- And perpetuity B

$t = 0$	1	2	3	4	5	6	…
0	0	0	0	C	C	C	…

Annuities

- Now consider a portfolio that earns the cash flows from perpetuity A, but has to pay the cashflows of perpetuity B.
- Let's draw the timeline of payments



- These are exactly the payments of the annuity!
- Two portfolios with the same cashflows must have the same price. The PV of the perpetuity must be the PV of perpetuity A, minus the PV of perpetuity B.
- The PV of this annuity is therefore

$$PV = \frac{C}{r} - \frac{C}{r}\frac{1}{(1+r)^3} = C\left[\frac{1-\frac{1}{(1+r)^3}}{r}\right]$$

Annuity: Example

- Which car can you afford? If
  - You have no large amount of cash
  - You can afford \$632 per month
  - You can borrow at an interest rate of 1% per month
  - You want to have paid the loan in full in 48 months

$$PV = c\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right]$$

$$= 632\left[\frac{1}{0.01} - \frac{1}{0.01(1+0.01)^{48}}\right] = 24{,}000$$

A Closer Look at the Frequency of Computing

- Earlier, we saw the power of compounding over long periods of time, but the effects of compounding also increase with the frequency
- Lenders are required by law to report the annual percentage rate (APR)
- APR = (rate per period)(# of periods per year)
- For example, a loan that charges 1% per month:
  APR = 1% * (12) = 12%
- Is 12% per year the "real" cost?
- How do you make a loan seem cheaper?

APR and EAR

- Effective annual rate (EAR): If interest is compounded *m* times a year, EAR is:

$$EAR = \left(1 + {}^{APR}/_{m}\right)^{m} - 1$$

- Which loan is the cheapest?
  - 10% compounded semi-annually
  - 10% compounded quarterly
  - 10% compounded daily

Continuous Compounding

- Consider increasingly frequent compounding: annually, quarterly, daily, every second,…
- When compounding happens "all the time," it is called continuous compounding
- What happens to the EAR?

$$EAR = e^{APR} - 1$$

- What happens to the FV?

$$FV = \lim_{N \to \infty} PV \times \left(1 + {}^{r}/_{N}\right)^{Nt} = PV \times e^{rt}$$

- Similarly, $PV = FVe^{-rt}$

Continuous vs Annual Compounding

- Suppose r=5% and PV-$1,000
- Annual compounding
  - 1 year?
  - 2 years?
  - T years?
- Continuous compounding
  - 1 year?
  - 2 years?
  - T years?

Example: Payday Loans

- Quick Cash offers 14-day loans to cash-strapped borrowers at a price of $117 per $100 lent. The loan is automatically rolled over at the same terms until repaid

- What APR is Quick Cash required to report?

- What is the effective annual rate of the advance?

Example: Payday Loans

- Quick Cash offers 14-day loans at $117 per $100 lent, rolled over until repaid
- What APR must Quick Cash report? What is the effective annual rate?
- Rate per 14 days = 17%. APR = 17% × 365/14 = 443%
- EAR = $1.17^{(365/14)} - 1 \approx 5{,}894\%$
- APR ignores compounding — here that is the whole cost

Summary

- A cash flow means nothing until you say when it arrives
- FV, PV, r and t: given any three, solve for the fourth
- Perpetuities and annuities are shortcuts, not new ideas
- APR is a quote; EAR is what you actually pay
- Next class: Introduction to Bonds (BMAE 3.1) — a bond is a cash flow stream