Foundations of Finance

Portfolio Theory I

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Today's Agenda

- Portfolios and portfolio weights
- Portfolio expected return and variance
- Diversification with two risky assets
- Why does diversification work?
- The investment opportunity set and the efficient frontier
- Perfect positive and negative correlation
- Optimal portfolio choice with two risky assets

Partial Review of the Last Class

- What is a volatility drag?
- What is the difference between arithmetic and geometric returns?
- What is covariance? What is positive and negative covariance?

Portfolio

- A combination of N assets, with returns R_1, R_2, R_3…….,R_N
- Portfolio *p*, with portfolio weights w_1, w_2, w_3,…..w_N
  - W_i is the wealth invested in asset *i*.

$$w_i = \frac{\$\ value\ of\ stock\ i's\ position}{total\ \$\ value\ of\ the\ portfolio}$$

  - Portfolio weights sum to one: w_1+ w_2+w_3+…..+w_N=1
  - A negative weight indicates a short position

Examples-Portfolio Weights

- Calculate portfolio weights for the following portfolios
- 1. A portfolio that invests in:
  - Stock A: 2,000 shares at $40
  - Stock B: 6,000 shares at $20
- 2. A portfolio that invests in:
  - Stock A: 2,000 shares at $40
  - Stock B: 6,000 shares at $20
  - Financed by $100,000 of own funds, $100,000 of borrowing
- 3. A portfolio that invests in
  - Short stock A: 2,000 shares at $40
  - Long stock B: 6,000 shares at $60

Portfolio Return and Portfolio Expected Return

- The return on the portfolio is

$$R_p = \sum_{i=1}^{N} w_i R_i = w_1 R_1 + w_2 R_2 + \cdots + w_N R_N$$

- The **expected** return on the portfolio is

$$E(R_p) = \sum_{i=1}^{N} w_i E(R_i)$$

Computing Portfolio Variance

- Best way to compute portfolio variance: make a table.
- Row i, column j: $w_i w_j Cov(R_i R_j)$
- Then just add up all the cells in the table!
- Note: $Cov(R_i R_i) = Var(R_i)$

	$w_1 r_1$	$w_2 r_2$	$\cdots$	$w_n r_n$
$w_1 r_1$	$w_1^2 \sigma_1^2$	$w_1 w_2 \sigma_1$	$\cdots$	$w_1 w_n \sigma$
$w_2 r_2$	$w_1 w_2 \sigma_1$	$w_2^2 \sigma_2^2$	$\cdots$	$w_2 w_n \sigma$
$\cdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$w_n r_n$	$w_1 w_n \sigma$	$w_2 w_n \sigma$	$\cdots$	$w_n^2 \sigma_n^2$

Portfolio Variance and SD

- With two securities, the portfolio variance is:

$$Var(R_p) = Var(w_1 R_1 + w_2 R_2)$$
$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \boxed{2w_1 w_2 Cov(R_1, R_2)}$$
$$= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + \boxed{2w_1 w_2 \rho_{12}\sigma_1\sigma_2}$$

- Note the covariance term! Without exaggeration this term is the key to understanding portfolio choice.
- In general, the portfolio variance is:

$$Var(R_p) = Var(w_1 R_1 + \cdots + w_N R_N)$$
$$= \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j>i}^{N} w_i w_j \rho_{ij}\sigma_i\sigma_j$$

- The standard deviation of the portfolio is: $\sigma_P = \sqrt{\sigma_P^2}$

Example: Diversification with 2 Risky Assets

- Suppose we have two risky assets, US and JP, with:

Asset	E[R]	σ
US	$E[R_1]=13.6\%$	$\sigma_1 = 15.4\%$
JP	$E[R_2]=15\%$	$\sigma_2 = 23.0\%$

- The correlation $\rho_{12} = 27\%$.
- If an investor holds $w_1 = 60\%$ in the US and $w_2 = 40\%$ in JP, what is the mean and volatility of the portfolio?

Example continued…

- Portfolio mean:

$$E(R_p) = 0.6*0.136 + 0.4*0.150 = 14.2\%$$

- Portfolio variance:

$$Var(R_p) = (0.6)^2*(0.154)^2 + (0.4)^2*(0.230)^2$$
$$+2*0.6*0.4*0.27*0.154*0.230 = 0.022$$

$$\sigma_p = 14.7\%$$

Diversification

- This portfolio has higher expected return and lower risk than holing US alone
- Illustrates the gains from diversification

Asset	E[R]	σ
US	$E[R_1]=13.6\%$	$\sigma_1 = 15.4\%$
JP	$E[R_2]=15\%$	$\sigma_2 = 23.0\%$
Portfolio	$E[R_2]=14.2\%$	$\sigma_p = 14.7\%$

Why Does Diversification Work?

- Think about betting $1 on a single coin flip
  - If heads, win $1 (50%)
  - If tails, lose $1 (50%)
- Now think about betting $0.50 on two separate coin flips
  - If HH, win $1 (25%)
  - If HT or TH, break even (50%)
  - If TT, lose $1 (25%)
- Diversifying between two coin flips is less risky!
- Half the time when one outcome is bad, the other outcome is good.

Why Does Diversification Work?

- Stocks are not coin flips
  - Returns can take any value, not binary • But same logic still applies.
- Consider investing in two stocks with uncorrelated returns
  - Then when Stock A is bad, Stock B can make up for it.
  - Let's say Stock A has return 2% lower than usual.
  - When will portfolio of A + B have more extreme return than Stock A?
  - Only when Stock B has return at least 2% lower than usual.
  - This happens less than half the time! Otherwise, portfolio more stable

Risk and Return with Varying Weights

- Let w be the weight in the US, and (1-w) the weight in JP
- The expected return on the portfolio is:

$$E[R_p] = w \times 0.136 + (1 - w) \times 0.150$$

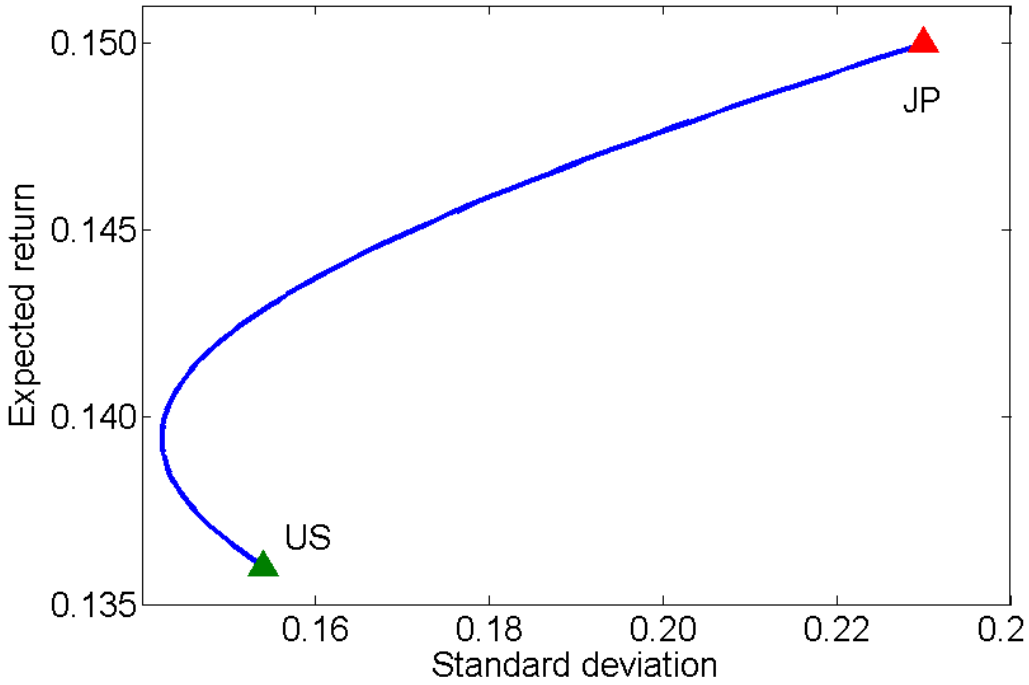
- The variance of the portfolio return is:

$$Var(R_p) = w^2 \times (0.154)^2 + (1 - w)^2 \times (0.230)^2 + 2 \times w \times (1 - w) \times 0.27 \times 0.154 \times 0.230$$

- What happens when we vary weight w?

Varying Portfolio Weights Gives: The Investment Opportunity Set

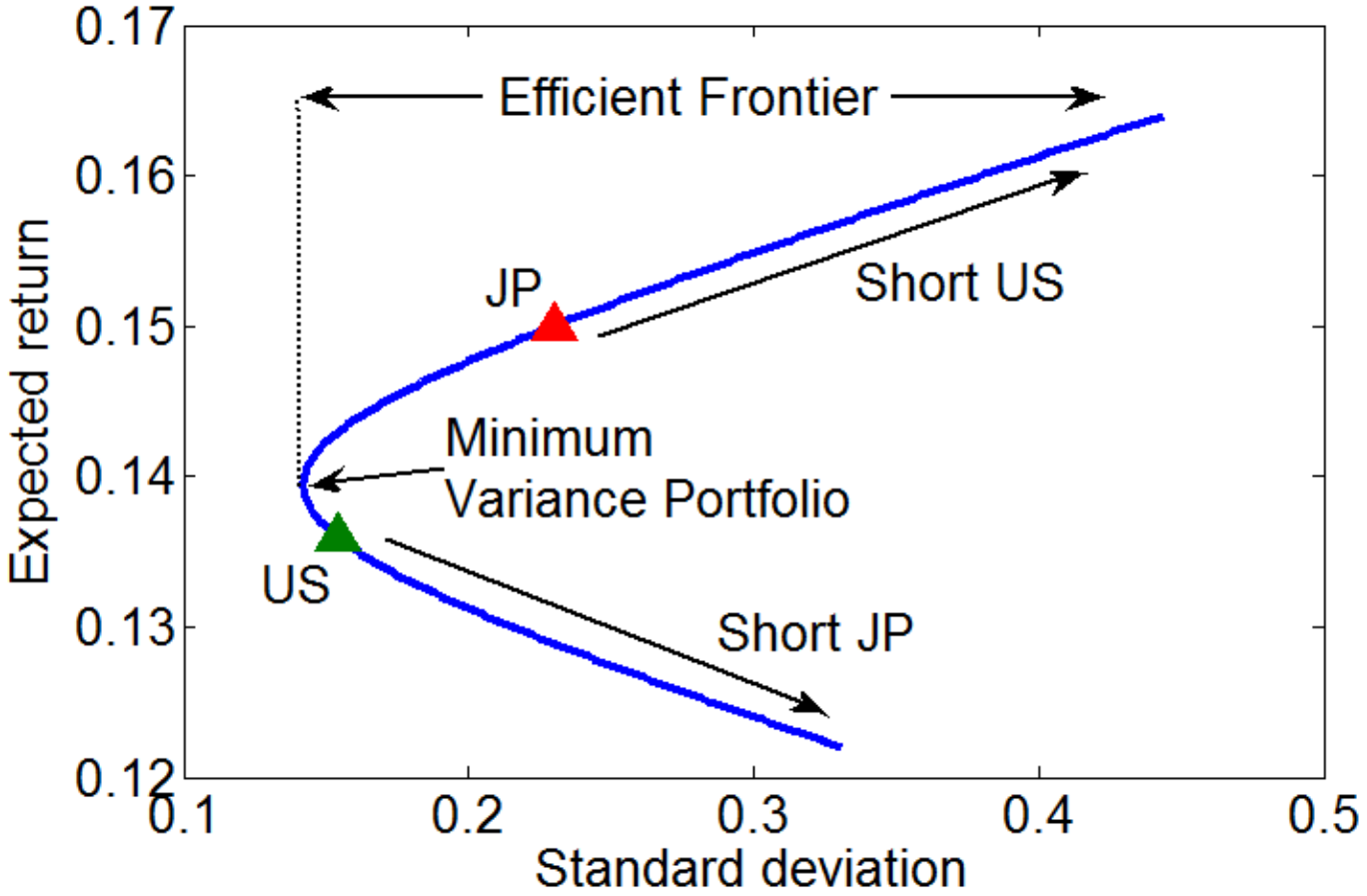
US Weight w	Mean $E[R_P]$	Volatility σ_P
0.0	0.150	0.230
0.1	0.149	0.212
0.2	0.147	0.195
0.3	0.146	0.179
0.4	0.144	0.166
0.5	0.143	0.155
0.6	0.142	0.147
0.7	0.140	0.143
0.8	0.139	0.143
0.9	0.137	0.146
1.0	0.136	0.154



Portfolio Terminology

- The investment opportunity set consists of all available risk-return combinations
- An efficient portfolio is a portfolio that has the highest possible expected return for a given standard deviation
- The efficient frontier is the set of efficient portfolios. It is the upper portion of the minimum variance frontier starting at the minimum variance portfolio
- The minimum variance portfolio (mvp) is the portfolio that provides the lowest variance (SD) among all possible portfolios of risky assets

Efficient Frontier



Perfect Positive and Negative Correlation

- Let's consider two special cases: perfect positive and negative correlation.
- Under perfect correlation, $\rho_{12} = 1$ and this reduces to

$$Var[R_p] = (w_1)^2 (\sigma_1)^2 + (w_2)^2 (\sigma_2)^2 + 2\, w_1\, w_2\, \sigma_1\, \sigma_2 = (w_1\, \sigma_1 + w_2\, \sigma_2)^2$$

- Because $\sigma_p = w_1\sigma_1 + w_2\sigma_2$, there are no gains from diversification.
- Under perfect negative correlation, $\rho_{12} = -1$ this reduces to

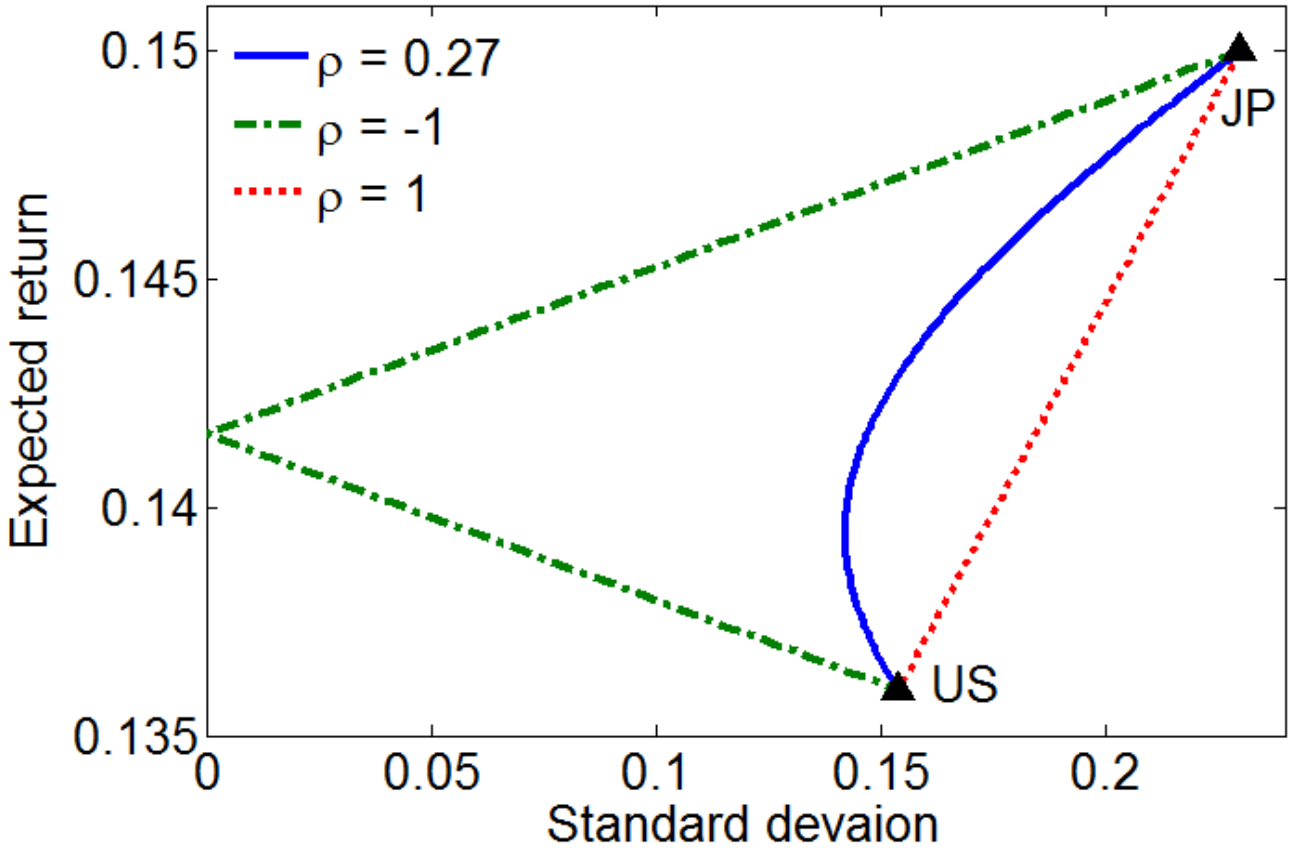
$$Var[R_p] = (w_1)^2 (\sigma_1)^2 + (w_2)^2 (\sigma_2)^2 - 2\, w_1\, w_2\, \sigma_1\, \sigma_2 = (w_1\, \sigma_1 - w_2\, \sigma_2)^2$$

- Here, $\sigma_p = |w_1\sigma_1 - w_2\sigma_2|$ (absolute value). The gains from diversification are maximal.

Zero Risk Portfolio with Perfect Negative Correlation

- Assume that the correlation between the US and the Japanese stock markets is -1.
- Determine the portfolio weights in Japan and the US that create a zero-risk portfolio
- Hint: recall that $w_1 + w_2 = 1$

Investment Opportunity Set with Varying Correlations



Optimal Portfolio Choice with Two Risky Assets

- Any (mean-variance) investor should choose an *efficient* portfolio to benefit from diversification
- Which efficient portfolio is *optimal* depends on investor's preferences, in particular her risk aversion
- We turn to modelling investor preferences next