Foundations of Finance

Portfolio Theory II

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Today's Agenda

- The risk-free asset
- A risky and a risk-free asset: the Capital Allocation Line
- The Sharpe ratio
- Two risky assets and a risk-free asset: the tangency portfolio
- Two-fund separation
- Practice questions
- Many risky assets: the portfolio optimizer

Partial Review of the Last Class

- Define:
  - Investment opportunity set
  - Efficient portfolio
  - Efficient frontier
  - Minimum variance portfolio
  - Optimal portfolio

Portfolio Selection with Riskless Security

- We have studied the two components of portfolio theory:
  - Feasible portfolios
  - Desirable portfolios
- Now, let's build on this and add more assets.
  - Portfolio selection with a risk-free and one risky security.
  - Portfolio selection with a risk-free and two risky securities.
    - Two-Fund Separation Theorem of Harry Markowitz.
  - Portfolio selection with a risk-free and many risky securities.
  - Systematic and idiosyncratic risk. The Single Index Model.

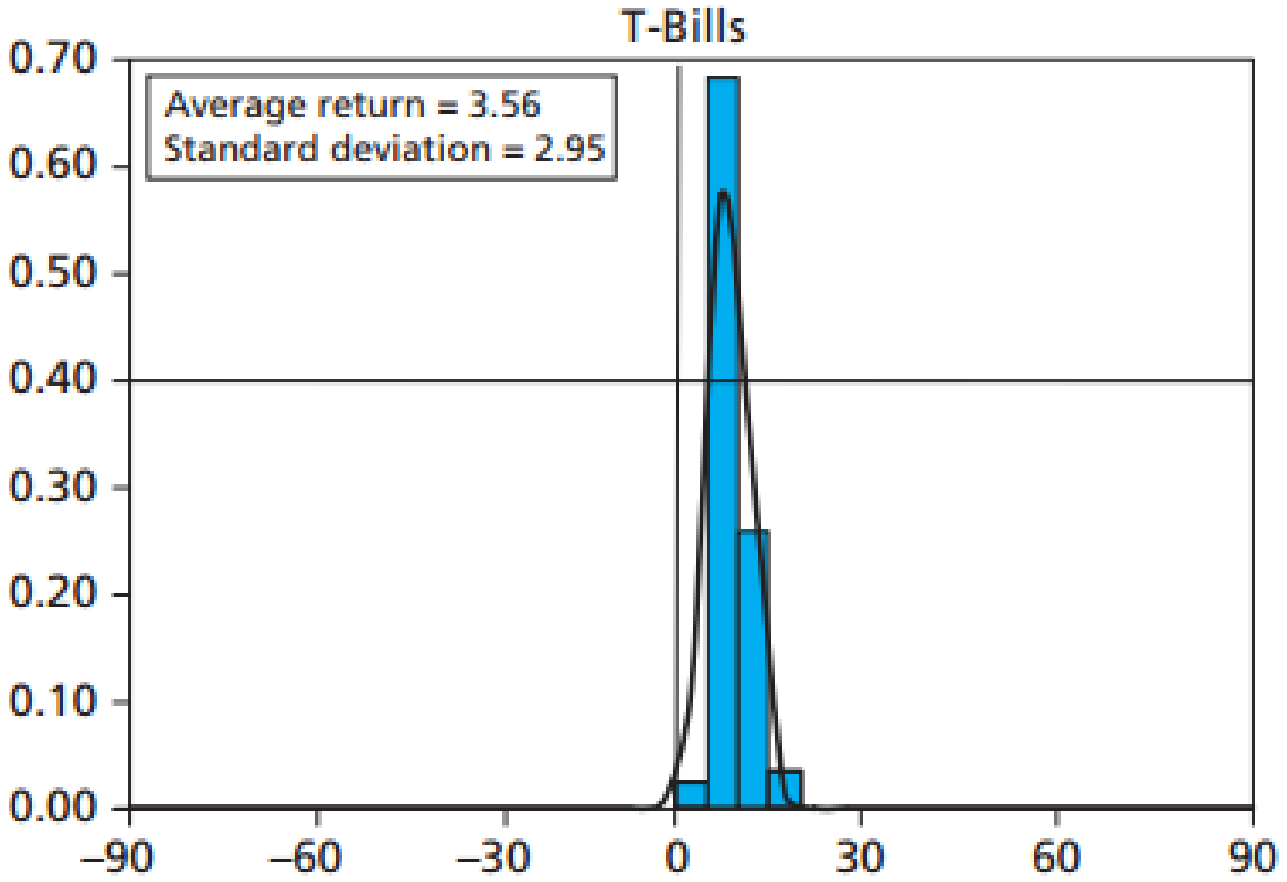
Properties of a Risk-Free Asset

- The risk-free return is denoted R_f
- The risk-free return is known for sure:

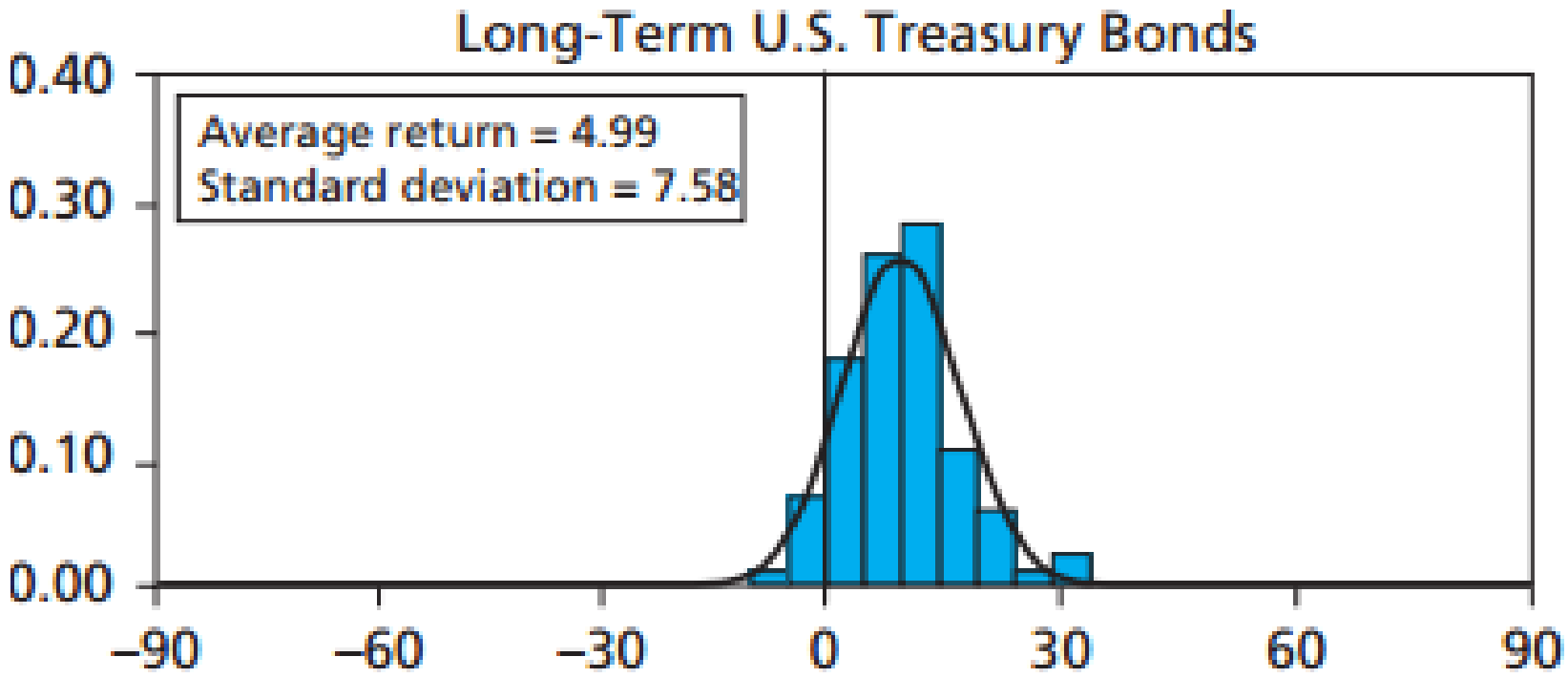
$$E[R_f] = R_f$$
$$Var[R_f] = \sigma_f^2 = 0$$
$$Cov(R_f, R_i) = 0 \; for \; any \; asset \; i$$

- The risk-free return is constant over time
- What is the best proxy for it?

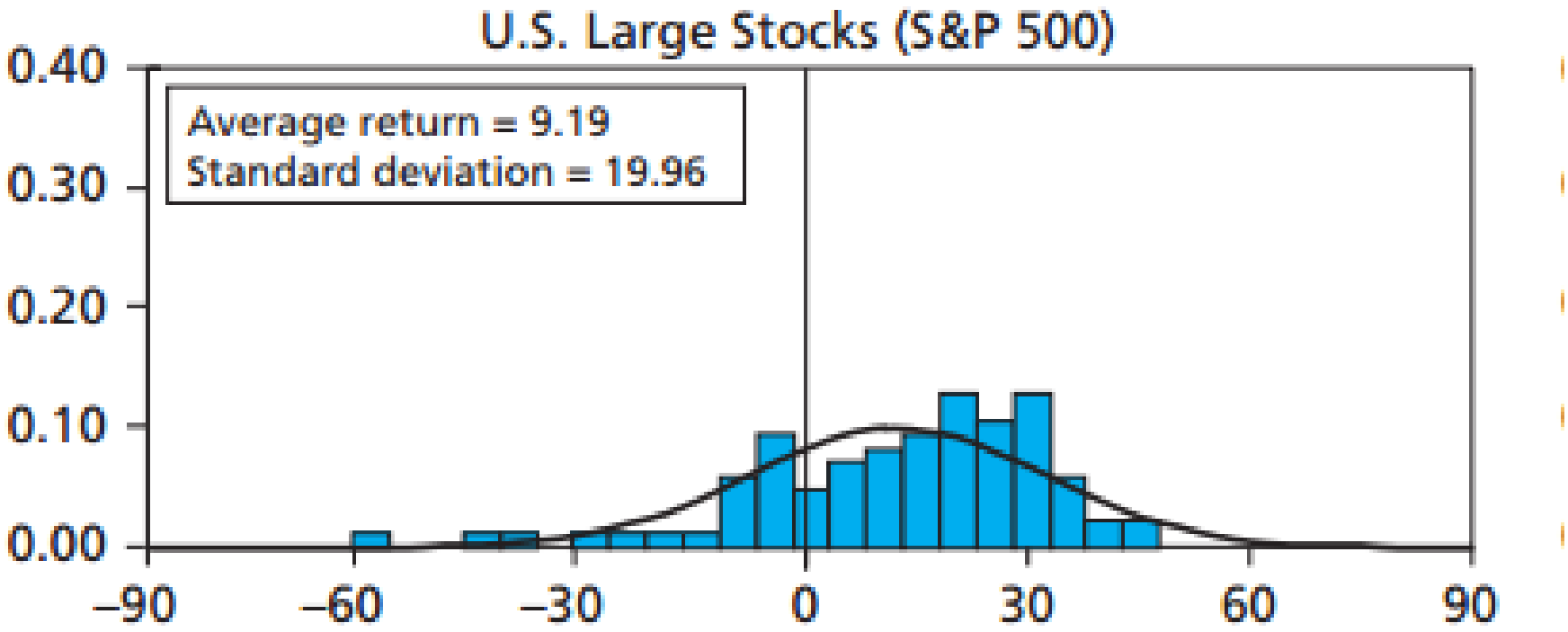
T-Bills Mean and SD



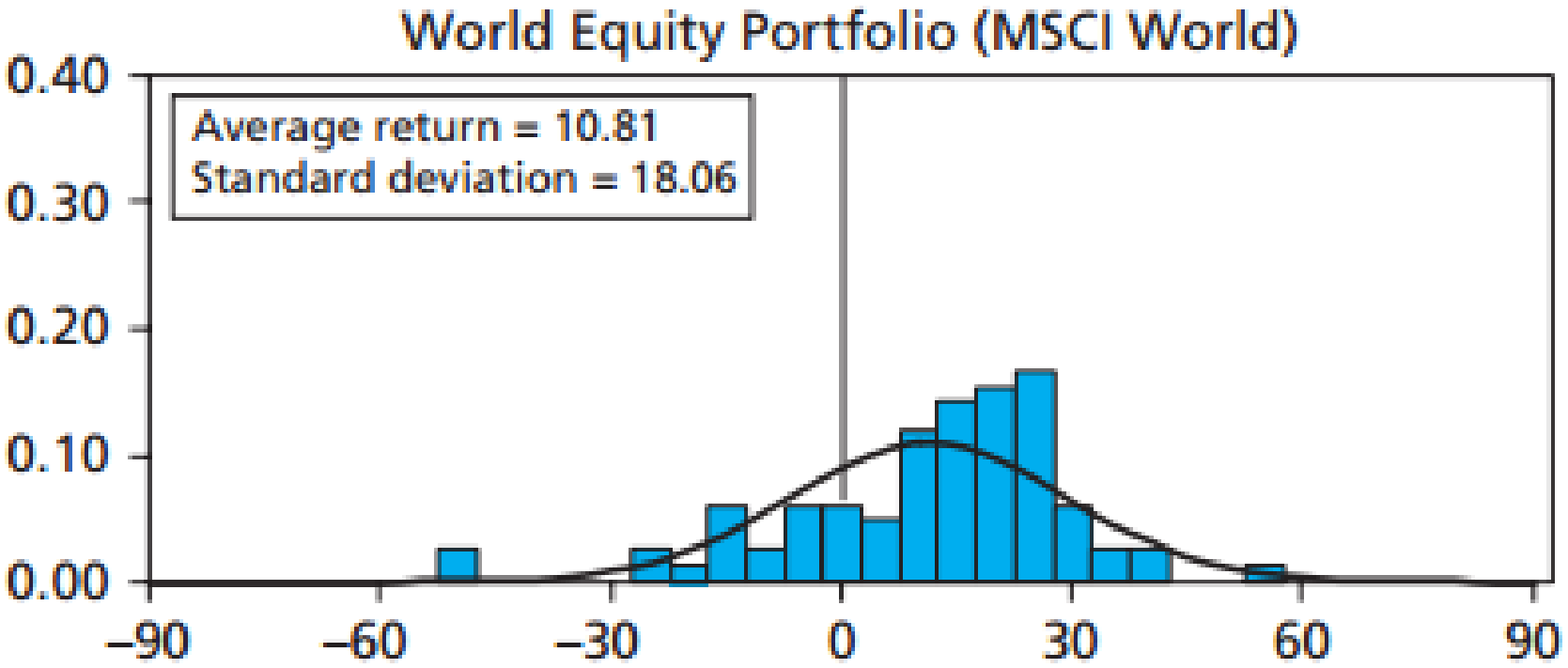
Long-Term US Treasury Bonds



US Large Stocks



World Equity Portfolio



Portfolio With a Risky and a Risk-free Asset

- Let w be the fraction of wealth invested in the risky asset and the rest in invested in the risk-free asset

- Expected portfolio return:

$$E[R_P] = wE[R_i] + (1-w)R_f = R_f + wE\underbrace{[R_i - R_f]}_{\text{risk premium}}$$

- Variance of a portfolio return:

$$\sigma_P^2 = w^2\sigma_i^2 + (1-w)^2\sigma_f^2 + 2w(1-w)\sigma_i\sigma_f\rho_{if} = w^2\sigma_i^2$$

- The standard deviation is:

$$\sigma_P = \sqrt{w^2\sigma_i^2} = |w|\sigma_i$$

Investment Opportunity Set with Risk-free and Risky Asset

- Consider various portfolios p (which are long in the risky asset and long or short in the risk free asset)

- What is the risk-return relationship? Let's combine the expected return and volatility formula from the last slide

$$E[R_P] = R_f + \frac{E[R_i - R_f]}{\sigma_i}\sigma_P$$
$$= R_f + (Sharpe\ Ratio\ of\ i)\sigma_P$$
$$= R_f + SR_i \times \sigma_P$$

- The last equation is called the **Capital Allocation Line**

Sharpe Ratio

- The last equation contains a term with special role in finance:

$$\frac{E[R_i - R_f]}{\sigma_i}$$

- A ratio developed by Noble laureate William F. Sharpe to measure risk-adjusted performance

- Calculated by subtracting the risk-free rate from the rate of return for a portfolio and dividing the result by the standard deviation of the portfolio returns

Sharpe Ratio of US Stock Market

- Example:
  - Risky asset: US Stock market
  - $E[R_{US}]=12\%$, $\sigma_{US}=20\%$
  - Risk-free asset: US T-Bill
  - $R_f=3\%$
- Hence,

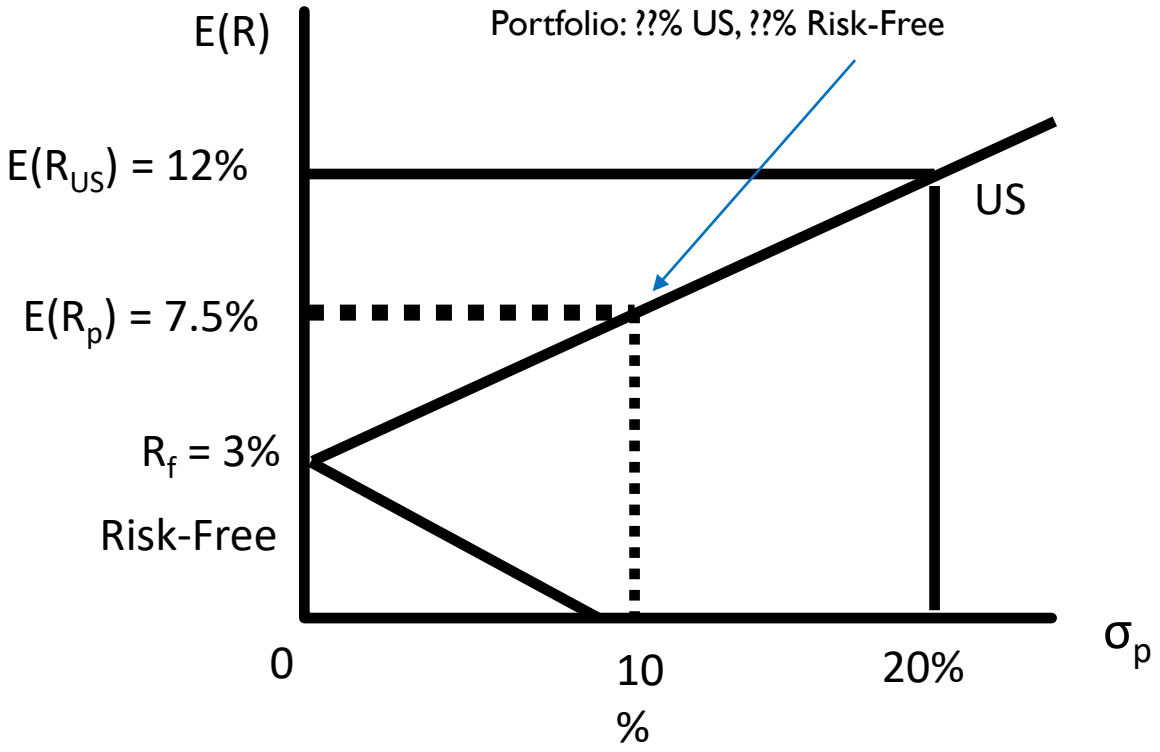
$$E[R_p] = R_f + SR \cdot \sigma_p$$

$$SR = \frac{E[R_{US}] - R_f}{\sigma_{US}} = \frac{0.12 - 0.03}{0.2} = 0.45$$

- Sharpe ratio=return premium per unit of risk or price of risk

Investment opportunity set: The Capital Allocation Line

- The slope of the capital allocation line is given by the Sharpe ratio



Example: The Capital Allocation Line

- In the previous example:
  - How can I get a 15.6% return by investing in US stocks and the risk-free asset?
  - What is the standard deviation of this portfolio?
  - Can you do this is the real world?

Example: The Capital Allocation Line

- In the previous example:
  - How can I get a 15.6% return by investing in US stocks and the risk-free asset?
    - Take on more risk than the US stock market
    - Short sell risk free asset = You borrow money
  - What is the standard deviation of this portfolio?
  - $E(R_p) = 3\% + 1.4*(12\% - 3\%) = 15.6\%$, $\sigma_p = 1.4*20 = 28\%$:
  - Can you do this is the real world?
  - Yes! It is called buying stock on margin.

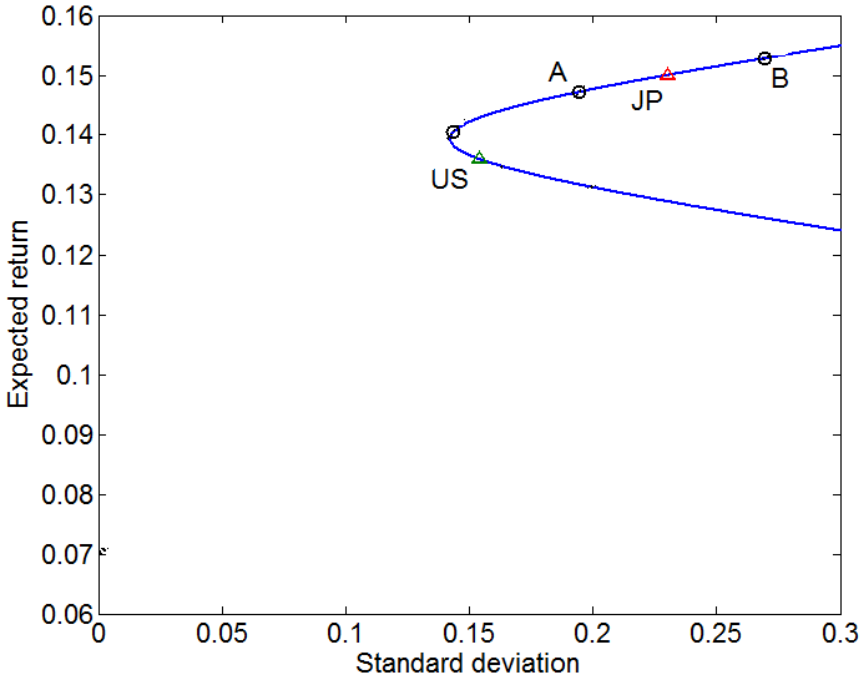
Optimal Portfolio selection with two risky and one risk-free asset

- Create the set of possible mean-SD combinations from different portfolios of risky assets.
- Now consider combination of the risk-free asset, with any other portfolio in the investment opportunity set.
- Find the "tangency portfolio," that is, the portfolio with the highest Sharpe ratio

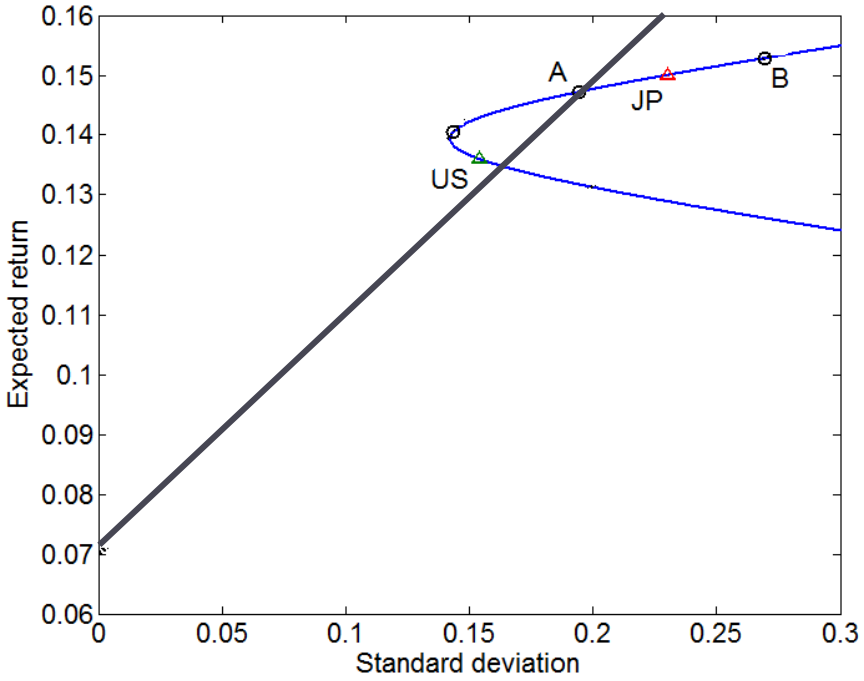
$$SR_i = \frac{E[R_i] - R_f}{\sigma_i}$$

- Choose the combination of the tangency portfolio and the risk free asset to suit your risk-return preferences.

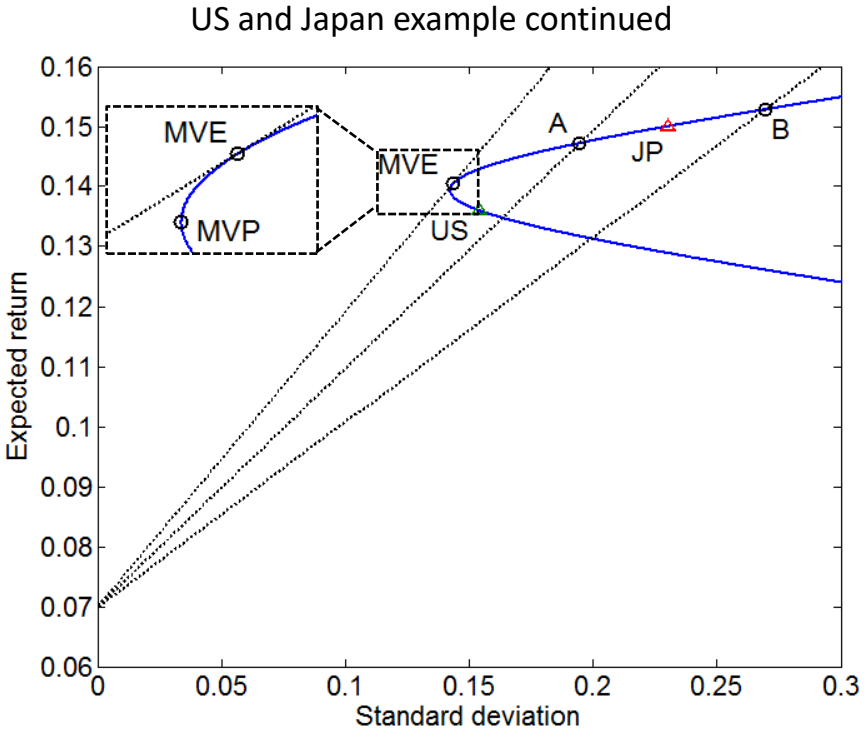
Two risky and one risk-free asset



Two risky and one risk-free asset



Two risky and One Risk-free Asset



- The tangency portfolio (Mean Variance Efficient MVE) maximizes the Sharpe ratio (slope): $$SR_i = \frac{E[R_i] - R_f}{\sigma_i}$$

Two-fund Separation

- This simple calculation suggests a powerful result about optimal portfolios
- All investors hold combinations of the same two "mutual funds"
  - The risk-free asset
  - The tangency portfolio
- An investor's risk aversion determines the fraction of wealth invested in the risk-free asset
- But, all investors should have the rest of their wealth invested in the tangency portfolio

Portfolio Choice-Terms

- Let's recap the key terms we have introduced.
  - The investment opportunity set.
  - An efficient portfolio.
  - The efficient frontier.
  - The minimum variance portfolio (MVP).
  - The mean-variance efficient portfolio (MVE).
  - The indifference curve for an investor.
  - The optimal portfolio for an investor.

Practice Question

- Suppose that among the many stocks in the market there are two securities, A and B, with the following characteristics: A has mean .08 and σ = .4 and B has mean .13 and σ = .6. If the correlation between these two is $\rho = -1$, and if it is possible to borrow and lend at the risk-free rate, R_f, then what must be the equilibrium risk-free rate?

Practice Question

- Consider portfolios with positions in the US and Indian equity markets. The (annual) expected return and standard deviation of returns for the 2 markets are as follows:

	US	India
E[r]	10%	22%
SD[r]	20%	35%

  The correlation between the returns is 0.2.

- In addition, assume that the (annual) risk-free (T-bill) rate is 5%.

Practice Question

- Calculate the expected returns and standard deviations of the following portfolios:
  - 50% in the risk-free asset, rest in 50% US and 50% India
  - 110% in the portfolio of 50% US and 50% India, finance by borrowing at risk-free rate
- What are the weights for investing in the risk-free asset and the portfolio that invests 50% in the US and 50% in India (the "risky asset") that produce same return as India? What is the expected standard deviation of that portfolio?

Practice Question

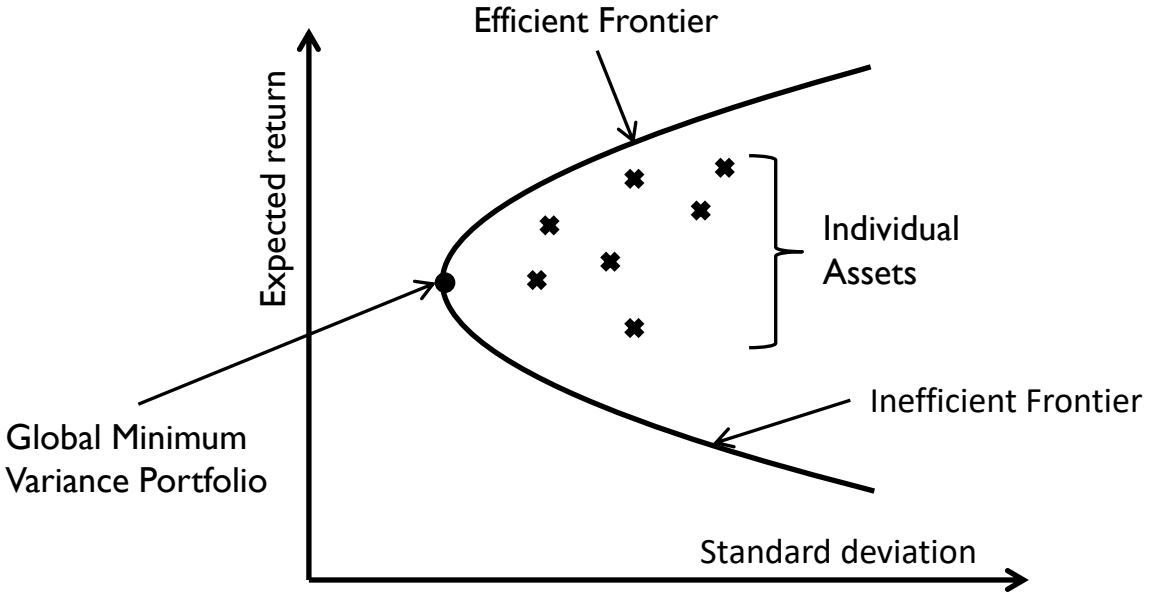
- The correlation coefficient between several pairs of stocks are as follows: Corr(A, B)=0.85, Corr(A, C)=0.6, Corr(A, D)=0.45. Each stock has an expected return of 8% and SD of 20%
- If your entire portfolio is now composed of stock A and you can add some of only one other stock to your portfolio, which one would you choose?
- Would your answer be different for more risk-averse or risk-tolerant investors?
- Suppose that in addition to investing in one more stock, you can invest in T-bills as well. Would you change your answer if the T-bill rate is 8%?

Portfolio Selection with Multiple Risky Assets

- Have covered optimal portfolios with two risky and one risk free asset.
- Two fund separation theorem.
- Next: turn to optimal portfolios with more than two risky assets.
- Next: optimal portfolios as number of assets grows large.

Investment Opportunity Set with Many Assets

- The first step in the analysis is to forget about the riskless asset and to identify the efficient portfolios of many risky assets alone.



Optimal Portfolio Selection with Many Risky Assets and a Risk-free Asset

- Now proceed just like with two risky assets, one risk-free asset.
- Create a set of possible mean-SD combinations from different portfolios of risky assets
- Find the 'tangency portfolio', i.e. the portfolio with the highest Sharpe ratio:

$$SR_i = \frac{E[R_i] - R_f}{\sigma_i}$$

- Choose the combination of the tangency portfolio and the risk-free asset to suit your risk-return preferences

Two-fund Separation

- Just like with two risky assets, the Two-Fund Separation theorem emerges.
- All investors hold combinations of the same two "mutual funds":
  - The risk-free asset
  - The tangency portfolio
- An investor's risk aversion determines the fraction of wealth invested in the risk-free asset.
- But, all investors should have the rest of their wealth invested in the tangency portfolio.

Application: Portfolio Optimizer (excel file)

- Calculates optimal portfolio with 5 risky assets and one riskless asset
- Why does the optimal portfolio load up on 2?
- Why hold asset 4 at all?
- Importance of correlation
  - $\rho_{4,5} = 0 \rightarrow 0.7$;
  - $\rho_{4,5} = 0.7 \rightarrow 0.9$